The Geometry of Quotient Varieties of Quivers

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Nederlandse Samenvatting

In deze thesis bestuderen we nader de representatietheorie van quivers. Aan een quiver Q (bestaande uit een verzameling punten V en pijlen A) en een dimensievector $\alpha : V \to \mathbb{N}$ kan men een variëteit van quiverrepresentaties $\mathsf{Rep}_\alpha Q$ associeren met een actie van een reductieve lineaire groep GL_α, die overeenkomt met basisverandering. Deze actie delen we uit om zo een nieuwe variëteit te bekomen, $\mathsf{iss}_\alpha Q$, die alle gesloten GL_α-banen in $\mathsf{Rep}_\alpha Q$ classificeert. Men kan zich dan afvragen hoe deze variëteit eruit ziet voor een gegeven quiver setting (Q, α), of omgekeerd aan welke voorwaarden een quiver setting moet voldoen opdat de variëteit $\mathsf{iss}_\alpha Q$ bepaalde eigenschappen heeft.

In het eerste hoofdstuk herhalen we de basisconcepten en methoden uit de algebraische meetkunde en de invariantentheorie, die we in de verdere hoofdstukken zullen gebruiken. Hoofdstuk twee is gewijd aan de vraag voor welke quiver settings (Q, α) de quotientruimte $\mathsf{iss}_\alpha Q$ de structuur heeft van een gladde variëteit. Zulke settings noemen we coregulier. We geven een reductiemethode voor quivers die de structuur van de quotientruimte $\mathsf{iss}_\alpha Q$ grotendeels invariant laat (d.w.z op een product met een affiene ruimte na). Daarmee bewijzen we dan dat elke coreguliere quiver setting te reduceren is tot een drietal eenvoudige basissettings.

In het derde hoofdstuk tonen we aan hoe men de coregulariteitsresultaten kan gebruiken om na te gaan of bepaalde moduliruimten van quivers glad zijn. We bepalen ook alle dimensie vectoren $(a_1, \dots, a_p; b_1, \dots, b_q)$ waarvoor de overeenkomstige representatievariëteit van $\mathbb{Z}_p * \mathbb{Z}_q$ glad is.

Het laatste hoofdstuk behandelt het onderzoek dat ik samen met mijn promotor Lieven Le Bruyn gedaan heb naar de symplectische structuur die men op de quiveralgebra $\mathbb{C}\bar{Q}$ kan zetten. We tonen aan dat de variëteit van α-dimensionale semisimpele representatieklassen van de gedeformeerde preprojectieve algebra,

$$\Pi_\lambda := \mathbb{C}\bar{Q}/(\sum_{v \in V} \lambda_v v - \sum_{a \in A} aa^* - a^*a),$$

voor bepaalde α's te beschouwen is als een coadjoint orbit van een oneindig dimensionale Lie-algebra die men kan construeren uit de quiver.

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Summary

In this thesis we study the representation theory of quivers. To a quiver Q (consisting of a set of vertices V and a set of arrows A) and a dimension vector $\alpha : V \to \mathbb{N}$ we can associate a variety of representations $\mathsf{Rep}_\alpha Q$ equipped with an action of a reductive linear group GL_α, corresponding to base change. If we divide out this action, we obtain a new variety, $\mathsf{iss}_\alpha Q$, classifying all closed orbits in $\mathsf{Rep}_\alpha Q$. One can wonder how this variety looks like for a given quiver setting (Q, α).

In the first chapter we repeat the basic concepts and methods from algebraic geometry and invariant theory. Chapter two is devoted to the question for which quiver settings (Q, α) the quotient space $\mathsf{iss}_\alpha Q$ is smooth. Such settings will be called coregular. We give a reduction method for quivers which keeps the structure of the quotient space $\mathsf{iss}_\alpha Q$ more of less invariant (i.e. up to a product with an affine space). We prove that every coregular quiver setting can be reduced to one of three basic settings.

In the third chapter we show how one can use this result to check whether certain moduli spaces are smooth. We apply these methods to find smooth representation spaces of $\mathbb{Z}_p * \mathbb{Z}_q$.

The last chapter treats the research on symplectic geometry which I did together with my promotor. We show that for certain α's the variety of α-dimensional semisimple representation classes of the deformed preprojectiv algebra,

$$\Pi_\lambda := \mathbb{C}\bar{Q}/(\sum_{v \in V} \lambda_v v - \sum_{a \in A} aa^* - a^* a),$$

can be seen as a coadjoint orbit of a infinite dimensional Lie-algebra associated to the quiver.

Chapter 1

Preliminaries

1.1 Geometrical Invariant theory

Suppose that $V \subset \mathbb{C}^n$ is a complex affine variety with an algebraic group G acting linearly on it. In general one is interested in the classification of the orbits in V. So one will try to construct a new variety V/G of which the points correspond to orbits in G and an algebraic projection map $\pi : V \to V/\mathsf{G}$ which is G-invariant. Because this projection map is continuous, it maps orbits whose closures intersect to the same point. Therefore, V/G can only classify the closed orbits in V.

In order to construct this variety one proceeds as follows. The action on V gives an action of automorphisms on the ring of polynomial functions on V, denoted by $\mathbb{C}[V]$. The ring $\mathbb{C}[V]$ has a subring, $\mathbb{C}[V]^{\mathsf{G}}$ consisting of the G-invariant functions. If this subring is finitely generated this ring will correspond to a new variety, which we denote by V/G. The natural embedding $\mathbb{C}[V]^{\mathsf{G}} \hookrightarrow \mathbb{C}[V]$ gives us a projection $V \to V/\mathsf{G}$.

It is however not always true that $\mathbb{C}[V]^{\mathsf{G}}$ is finitely generated. For a certain class of algebraic groups, the reductive groups, this condition always holds. A group G is called reductive if every finite-dimensional representation is semisimple. Examples of such groups are finite groups and the classical linear groups $\mathsf{GL}_n(\mathbb{C})$, $\mathsf{SL}_n(\mathbb{C})$, $\mathsf{O}_n(\mathbb{C})$, $\mathsf{SO}_n(\mathbb{C})$ and $\mathsf{Sp}_n(\mathbb{C})$. Moreover direct sums, normal subgroups and homomorphic images of reductive groups are reductive so we have a vast class of reductive groups.

Theorem 1.1 (The Algebraic Quotient). If G is a reductive group and V a variety with a linear action of G, the ring of invariants $\mathbb{C}[V]^{\mathsf{G}}$ is finitely generated and the corresponding variety is called the *algebraic quotient* and denoted by V/G. The projection $\pi : V \to V/\mathsf{G}$ is surjective and two points $x, y \in V$ have the same image if and only if the closures of their orbits $\bar{\mathcal{O}}_x$, $\bar{\mathcal{O}}_y$ intersect. Every fiber $\pi^{-1}(\xi)$ contains one unique closed orbit and therefore it is the best possible approximation of an orbit space. (For more information see [18, II.3.2])

Determining the structure of a quotient variety is in general not an easy problem. One can try first to look at the local structure of of V/G around a point ξ. To do that we use a very handy theorem by Luna [21], [24]. But we first have to introduce the concept of an étale map.

Definition 1.2. A morphism $\varphi : V \mapsto W$ between two varieties is called *étale* if it is smooth and the fibers are finite. In smooth points $x \in V$ this means that the differential

$$d_x\varphi : \mathrm{T}_x V \to \mathrm{T}_{\varphi(x)} W$$

is an isomorphism.

Suppose Y is a subvariety of X. An étale morphism $\phi : U \to X$ together with an injection $s : Y \hookrightarrow U$ is called a neighborhood of Y if the following diagram commutes.

$$\begin{tikzcd} & U \arrow[dr, "\varphi"] & \\ Y \arrow[ur, "s"] \arrow[r, hook] & \varphi(U) \arrow[r, hook] & X. \end{tikzcd}$$

This concept of neighborhoods allows us to define a topology in the sense of Grothendiek, intersections are given by fiberproducts

$$(U \to X) \cap (V \to X) = U \times_X V \to X.$$

The restriction of a morphism $\psi : X \to Y$ to an neighborhood $U \to X$ is given by the composition

$$U \to X \to Y.$$

A concise introduction to this subject is given in [24].

Suppose now that G is a reductive group, V is a G-variety and that we want to study V/G around a point $\xi \in V/\mathsf{G}$. Take now a point $p \in V$ such that the orbit $\mathcal{O}_p$ is the closed orbit corresponding to ξ. For the sake of simplicity we will assume that our variety V is smooth in p, for the general case we refer to [16].

To take a look at the local structure around ξ, we are going to use the Luna-slice theorem. This theorem connects the structure of V around the orbit of p under the action of G to the structure of V/G around the corresponding point ξ.

Denote the tangent space of the orbit $\mathcal{O}_p$ in p by $\mathrm{T}_p\mathcal{O}_p$. This tangent space is of course a subspace of the complete tangent space T_pV. We denote the quotient space by $\mathsf{N}_p := \mathrm{T}_p V / \mathrm{T}_p \mathcal{O}_p$. Now the stabilizer of p in G,

$$\mathsf{Stab}_p := \{g \in \mathsf{G} | gp = p\},$$

has an induced action on both $\mathrm{T}_p V$ and $\mathrm{T}_p \mathcal{O}_p$ and hence also one on N_p. This enables us to consider a fiberbundle

$$\mathsf{G} \times^{\mathsf{Stab}_p} \mathsf{N}_p := (\mathsf{G} \times \mathsf{N}_p)/\mathsf{Stab}_p,\ \forall h \in \mathsf{Stab}_p : (g, n)^h = (gh^{-1}, hn)$$

This fiber bundle is also a G-variety by left multiplication on the G factor. The first part of the slice theorem indicates the connection between this fiber bundle and V.

Theorem 1.3 (Luna Slice part 1). There exists an étale morphism φ between an open neighborhood of the point $(1, 0) \in \mathsf{G} \times^{\mathsf{Stab}_p} \mathsf{N}_p$ and an open neighborhood of $p \in V$. This étale map maps point of $\mathsf{G} \times^{\mathsf{Stab}_p} 0$ onto $\mathcal{O}_p$.

Both of the spaces, $\mathsf{G} \times^{\mathsf{Stab}_p} \mathsf{N}_p$ and V have a left action of G on it, so we can make the quotient on both sides. The quotient of the fiber bundle has a fairly simple structure because the action eats up the G part of the product, leaving us with the quotient space $\mathsf{N}_p/\mathsf{Stab}_p$. The quotient of V is already known and is simply V/G. The second part of the Luna slice theorem gives us the connection between the two quotient spaces.

Theorem 1.4 (Luna Slice part 2). The étale isomorphism of part 1 is G-equivariant. Dividing out the action of G gives us the following commutative diagram, in which the map φ/G is also an étale morphism between an open neighborhood of the point $0 \in \mathsf{N}_p/\mathsf{Stab}_p$ and an open neighborhood of $\xi \in V/\mathsf{G}$.

$$\begin{tikzcd}
\mathsf{G} \times^{\mathsf{Stab}_p} \mathsf{N}_p \arrow[r, "\varphi"] \arrow[d, "/\mathsf{G}"'] & V \arrow[d, "/\mathsf{G}"'] \\
\mathsf{N}_p/\mathsf{Stab}_p \arrow[r, "\varphi/\mathsf{G}"] & V/\mathsf{G}
\end{tikzcd}$$

Proofs of these theorems and further information can be found in [24] and [21].

Because there is an étale morphism between $\mathsf{N}_p/\mathsf{Stab}_p$ and V/G the implicit function theorem states that there are analytical neighborhoods (i.e. with respect to the standard topology on $\mathbb{C}$)

$$0 \in U \subset \mathsf{N}_p/\mathsf{Stab}_p,\ \xi \in U' \subset V/\mathsf{G}$$

that are analytically isomorphic. Many properties like smoothness are analytically invariant and hence the Luna slice theorem now states that to investigate

V/G around ξ one can as well look at the space $\mathsf{N}_p/\mathsf{Stab}_p$ around the zero. This has two main advantages. First of all we go from a big group G and a big variety V to a smaller group, namely a subgroup of G, and a smaller variety. Secondly the new variety is a vector space, so the action has presumably a simpler structure than the original. To use this fact we will finish this section by taking a closer look to linear group actions on vector spaces.

If our variety V is a vector space and the action of G is linear, $\mathbb{C}[V]$ is a polynomial ring and we can equipe this ring with the standard gradation. Because of the linearity of the action, on V the corresponding action on $\mathbb{C}[V]$ preserves the gradation. As a consequence the subring of invariant functions is again a graded algebra. This implies that the corresponding variety V/G is a cone through the zero with the projective variety $\mathsf{Proj}\ \mathbb{C}[V]^{\mathsf{G}}$ at infinity. The only possibiblity to assure that such a cone is a smooth variety, is that $\mathsf{Proj}\ \mathbb{C}[V]^{\mathsf{G}}$ is a projective space. This implies that V/G is an affine space and hence $\mathbb{C}[V]^{\mathsf{G}}$ is a polynomial ring. This observation translates to the following theorem:

Theorem 1.5. Suppose V is a vector space with a linear action of a reductive group G. V/G is smooth in the point corresponding to the zero orbit if and only if $V/\mathsf{G} \cong \mathbb{C}^t$ for a $t \in \mathbb{N}$. (See also [18, II.4.3 lemma 1])

This theorem is very useful to find singularities in quotient varieties. Take any smooth point p in a G-variety with closed orbit. To determine whether its corresponding point $\xi \in V/\mathsf{G}$ is singular or not, one has to look at the quotient variety $\mathsf{N}_p/\mathsf{Stab}_p$. If ξ is a smooth point, $\mathsf{N}_p/\mathsf{Stab}_p$ is smooth in the zero and it is thus an affine space. So whenever one finds a singularity in $\mathsf{N}_p/\mathsf{Stab}_p$ (not necessarely in the zero) $\mathsf{N}_p/\mathsf{Stab}_p$ is singular in the zero and the same holds for $\xi \in V/\mathsf{G}$. This method will be frequently used in what follows.

1.2 Representation theory

We can now apply the methods of the previous section to the special case of representations of an algebra. We follow [18, II.2.7].

Suppose that A is a finitely generated algebra with generators $a_1, \ldots, a_k$. One can consider the set of all n-dimensional complex representations

$$\mathsf{Rep}_n A := \{\rho : A \to \mathsf{Mat}_n(\mathbb{C})\}$$

as an algebraic subset of the affine space $\mathsf{Mat}_n(\mathbb{C})^k \cong \mathbb{C}^{n^2k}$ via the identification

$$\mathsf{Rep}_n A \hookrightarrow \mathsf{Mat}_n(\mathbb{C})^k : \rho \mapsto (\rho(a_1), \ldots, \rho(a_k)).$$

On this space there is natural action of $\mathsf{GL}_n(\mathbb{C})$ by conjugation. The orbits of this action correspond to the isomorphism classes of n-dimensional representations.

If we want to make the algebraic quotient, we first have to determine what kind of isomorphism classes correspond to closed orbits.

With every representation ρ one can also naturally associate an n-dimensionial A-module and vice versa:

$$M_\rho = \mathbb{C}^n \text{ and } \forall a \in A : \forall m \in M_\rho : a \cdot m := \rho(a)m.$$

The n-dimensional modules and representations are in fact 2 two ways of looking at the same structure so for every representation ρ, M_ρ will denote its corresponding module.

We recall that a representation ρ is called *simple* if its module M_ρ has no nontrivial submodule. Direct sums of simple modules are called *semisimple* and representations whose module cannot be written as a direct sum of lower dimensional modules are indecomposible. Every representation ρ has a decomposition row

$$0 = M_{\rho_1} \subset \cdots \subset M_{\rho_m} = M_\rho$$

of which the Jordan-Hölder factors $M_{\rho_{k+1}}/M_{\rho_k}$ are simple and uniquely determined. Keeping this in mind we can formulate the correspondence between geometrical orbits and isomorphism classes.

Theorem 1.6. An n-dimensional representation ρ is semisimple if and only if its corresponding orbit $\mathcal{O}_\rho$ is closed in $\mathsf{Rep}_n A$. If $\mathcal{O}_\rho$ is not closed the unique closed orbit $\mathcal{O}_{\rho'}$ contained in the closure of $\mathcal{O}_\rho$ will correspond to the isomorphism class of the direct sum of the Jordan-Hölder factors of ρ.

The quotient variety $\mathsf{Rep}_n A/\mathsf{GL}_n(\mathbb{C})$ will classify all n-dimensional isomorphism classes of semisimple representations. We will denote this variety as $\mathsf{iss}_n A$.

To apply the Luna slice theorem in the case of the variety $\mathsf{iss}_n A$ we have to work out what kind of structure the stabilizer and the normal space have in the case of a representation.

For a general representation ρ, the stabilizer Stab_ρ can be seen als the module automorphisms of M_ρ. A map $g \in \mathsf{GL}_n(\mathbb{C})$ is an element of $\mathsf{Aut}_A M_\rho$ if and only if

$$\forall a \in A : \forall m \in M_\rho : a \cdot gm = \rho(a)gm = g(a \cdot m) = g\rho(a)m.$$

Because this identity holds for all m we can say that this is the same as stating that $\forall a \in A : g^{-1}\rho(a)g = \rho(a)$ and we conclude that

$$\mathsf{Stab}_\rho \cong \mathsf{Aut}_A M_\rho.$$

Because we only need the structure of $\mathsf{Aut}_A M_\rho$ for semisimple representations it is natural to take first a look at the case of simple representations. This structure is well known by Schurs lemma:

Theorem 1.7. Suppose ρ and σ are simple representations, then we have the following identities.

1. $\mathsf{Aut}_A M_\rho = \mathsf{Aut}_A M_\sigma = \mathbb{C}^* =$ the scalar invertible matrices.
2. $\mathsf{Hom}_A(M_\rho, M_\sigma) = \begin{cases} 0 & \rho \ncong \sigma \\ \mathbb{C} & \rho \cong \sigma \end{cases}$

With this in mind we can now try to calculate the automorphism group of a general semisimple module. Suppose M is a semisimple module which has the following decomposition in simples

$$M \cong S_1^{\oplus e_1} \oplus \cdots \oplus S_k^{\oplus e_k}.$$

The vector space $\mathbb{C}^n$ on which this representation works can be written as

$$(\mathbb{C}^{n_1})^{\oplus e_1} \oplus \cdots \oplus (\mathbb{C}^{n_k})^{\oplus e_k}$$

where n_i is the dimension of S_i. We can decompose g as a direct sum of homomorphisms between the different components

$$g := \bigoplus_{1 \leq i,j \leq k} \bigoplus_{1 \leq r \leq e_i} \bigoplus_{1 \leq s \leq e_j} g^{ij}_{rs}$$

$g^{ij}_{rs}, i \neq j$ must be zero because there are no non-trivial homomorphisms between two different simples. For similar reasons g^{ii}_{rs} must be a scalar matrix, so we can write

$$g = \bigoplus_{1 \leq i \leq k} \bigoplus_{1 \leq s,t \leq e_i} \alpha^i_{rs} 1_{n_i},\ \alpha^i_{rs} \in \mathbb{C}$$

So for every $i = 1, \ldots, k$ g defines a $e_i \times e_i$-matrix α^i. On those matrices there is no extra restriction, and hence every set of k such matrices defines an endmorphism of M. It is easy to see that composition of endomorphisms is the same as multiplying the matrices so one can conclude:

Theorem 1.8. Suppose M is a semisimple A-module with decomposition

$$M \cong S_1^{\oplus e_1} \oplus \cdots \oplus S_k^{\oplus e_k}.$$

The endomorphism ring and the automorphism group of M are given by

1. $\mathsf{End}_A(M) \cong \bigoplus_{i=1}^k \mathsf{Mat}_{e_i \times e_i}(\mathbb{C})$
2. $\mathsf{Aut}_A(M) \cong \bigoplus_{i=1}^k \mathsf{GL}_{e_i}(\mathbb{C})$

An element of $\alpha^1 \oplus \cdots \oplus \alpha^k \in \mathsf{End}_A(M)$ works on $\mathbb{C}^n$ by multiplication with the matrix

$$(\alpha^1 \otimes 1_{n_1}) \oplus \cdots \oplus (\alpha^k \otimes 1_{n_k})$$

The second problem we have to attack is the structure of the normal subspace. In order to do that we first have to recall some representation-theoretical concepts.

Definition 1.9. If ρ and σ are two representations, an extension of σ by ρ is a short exact sequence

$$M_\rho \hookrightarrow M_\epsilon \twoheadrightarrow M_\sigma\ .$$

Two extensions are called equivalent if there is an isomorphism ϕ that makes the diagram below commutative

$$\begin{tikzcd} M_\rho \arrow[r, hook] \arrow[d, "1"] & M_\epsilon \arrow[r, two heads] \arrow[d, "\phi"] & M_\sigma \arrow[d, "1"] \\ M_\rho \arrow[r, hook] & M_\epsilon \arrow[r, two heads] & M_\sigma \end{tikzcd}$$

The set of extensions modulo this equivalence relation is denoted by $\mathsf{Ext}(\sigma, \rho)$.

Another look at extensions goes like this. If ρ is an n-dimensional representation and σ an m-dimensional representation, the exactness of an extension implies that representation corresponding to the middle term M_ϵ must be of the form

$$\begin{pmatrix} \rho & \lambda \\ 0 & \sigma \end{pmatrix}$$

where $\lambda : A \to \mathsf{Mat}_{n\times m}(\mathbb{C})$ is a linear map. In order to be a new representation we must have that

$$\forall a, b \in A : \begin{pmatrix} \rho(ab) & \lambda(ab) \\ 0 & \sigma(ab) \end{pmatrix} = \begin{pmatrix} \rho(a) & \lambda(a) \\ 0 & \sigma(a) \end{pmatrix} \begin{pmatrix} \rho(b) & \lambda(b) \\ 0 & \sigma(b) \end{pmatrix}$$

which means that $\lambda(ab) = \rho(a)\lambda(b) + \lambda(a)\sigma(b)$.

When do two such maps λ and λ' give equivalent extensions? Because of the structure of the commutative diagram one can write the automorphism as a matrix of the form below

$$\begin{tikzcd} M_\rho \arrow[r, hook] \arrow[d, "1"] & M_\epsilon \arrow[r, two heads] \arrow[d, "\phi"] & M_\sigma \arrow[d, "1"] \\ M_\rho \arrow[r, hook] & M_\epsilon \arrow[r, two heads] & M_\sigma \end{tikzcd} \Rightarrow \phi = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}, \zeta \in \mathsf{Mat}_{n\times m}(\mathbb{C}).$$

This gives us the following

$$\forall a, \in A : \begin{pmatrix} \rho(a) & \lambda'(a) \\ 0 & \sigma(a) \end{pmatrix} = \begin{pmatrix} 1 & -\zeta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \rho(a) & \lambda(a) \\ 0 & \sigma(a) \end{pmatrix} \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}$$

or $\lambda(a) - \lambda'(a) = \rho(a)\zeta - \zeta\sigma(a)$. Out of these calculations we can conclude:

Theorem 1.10. Consider the vectorspaces

- $Z(\sigma, \rho) := \{\lambda \in \mathsf{Hom}_{\mathbb{C}}(A, \mathsf{Mat}_{n\times m}(\mathbb{C}) | \lambda(ab) = \rho(a)\lambda(b) + \lambda(a)\sigma(b)\}$
- $B(\sigma, \rho) := \{\rho\zeta - \zeta\sigma | \zeta \in \mathsf{Mat}_{n\times m}(\mathbb{C})\}$.

Using these notations $\mathsf{Ext}(\sigma, \rho)$ can be considered as the quotient space

$$\frac{Z(\sigma, \rho)}{B(\sigma, \rho)}.$$

This theorem can be used to connect the tangent space of the representation variety with the space of extensions. Take $\rho \in \mathsf{Rep}_n A$, if $X \in \mathrm{T}_\rho \mathsf{Rep}_n A$ we must have that

$$\rho + \epsilon X : A \to \mathsf{Mat}_{n\times n}(\mathbb{C}[\epsilon]/(\epsilon^2))$$

is an algebra morphism, so

$$\rho(ab) + \epsilon X(ab) = \rho(a)\rho(b) + \epsilon(\rho(a)X(b) + X(a)\rho(b)) + \epsilon^2 X(a)X(b).$$

This is equivalent to saying that $X \in Z(\rho, \rho)$ and hence $\mathrm{T}_\rho \mathsf{Rep}_n A \subset Z(\rho, \rho)$.

The tangent space to the orbit of ρ, $\mathrm{T}_\rho \mathcal{O}_\rho$ can be identified with $B(\rho, \rho)$. To do this we use the surjective map

$$(d\mu)_e : \mathfrak{gl}_n(\mathbb{C}) \to \mathrm{T}_\rho \mathcal{O}_\rho : v \mapsto \frac{(1 + \epsilon v)\rho(1 - \epsilon v) - \rho}{\epsilon} = v\rho - \rho v.$$

whose image is indeed $B(\rho, \rho)$.

Theorem 1.11. For every $\rho \in \mathsf{Rep}_n A$ we have a natural injection

$$\mathsf{N}_\rho = \frac{\mathrm{T}_\rho \mathsf{Rep}_n A}{\mathrm{T}_\rho \mathcal{O}_\rho} \subset \mathsf{Ext}(\rho, \rho)$$

If $\mathsf{Rep}_n A$ is reduced and smooth in ρ this injection is a bijection.

Remark. If we look at this situation from the scheme-theoretic point this injection is always an isomorphism of spaces. The problem is however that if we only look at the tangent space as a variety we loose much information and our tangent space becomes smaller, only in the case when our point is smooth in the scheme theoretic sense the bijection remains the same as a variety. This is in fact the case when $\mathsf{Rep}_n A$ is reduced. More information can be found in [13].

The last element we need to apply the Luna-slice theorem is the action of Stab_ρ on N_ρ. We suppose again that ρ is a semisimple orbit with decomposition $S_1^{\oplus e_1} \oplus \cdots \oplus S_k^{\oplus e_k}$ and all notations the same as above. Suppose $\lambda \in \mathrm{T}_\rho \mathsf{Rep}_n A$ then we can write it out in its different component according to the decomposition of $\mathbb{C}^n$.

$$\lambda := \bigoplus_{1\leq i,j\leq k} \bigoplus_{1\leq s\leq e_i} \bigoplus_{1\leq t\leq e_j} \lambda_{st}^{ij}, \ \lambda_{st}^{ij} : A \to \mathsf{Mat}_{n_i \times n_j}$$

Each of the λ_{st}^{ij} is an extension S_i by S_j. The action of an element $\alpha^1 \oplus \cdots \oplus \alpha^k \in \mathsf{End}_A(M)$ works on λ by conjugation with the matrix

$$(\alpha^1 \otimes \mathsf{1}_{n_1}) \oplus \cdots \oplus (\alpha^k \otimes \mathsf{1}_{n_k})$$

this means that

$$\lambda_{st}^{ij} \mapsto (\alpha^i)_{su}^{-1} \lambda_{st}^{uv} (\alpha^j)_{vt}.$$

To simplify this description we introduce a new combinatorial tool.

1.3 Quivers

A *quiver* $Q = (V, A, s, t)$ consists of a set of vertices V, a set of arrows A between those vertices and maps $s, t : A \to V$ which assign to each arrow its starting and terminating vertex. We also denote this as

$$\textcircled{\scriptsize t(a)} \xleftarrow{a} \textcircled{\scriptsize s(a)}.$$

The *Euler form* of Q is the bilinear form $\chi_Q : \mathbb{Z}^{\#V} \times \mathbb{Z}^{\#V} \to \mathbb{Z}$ defined by the matrix

$$m_{ij} = \delta_{ij} - \#\{a | \textcircled{\scriptsize i} \xleftarrow{a} \textcircled{\scriptsize j}, \}$$

where δ is the Kronecker delta. A quiver $Q = (V, A, s, t)$ is *symmetric* if and only if the number of arrows between two vertices is the same in both directions, that is,

$$\forall v, w \in V : \#\{a \in A | \textcircled{\scriptsize v} \xrightarrow{a} \textcircled{\scriptsize w}\} = \#\{a \in A | \textcircled{\scriptsize v} \xleftarrow{a} \textcircled{\scriptsize w}\}.$$

or equivalently it's Euler form is symmetric.

A sequence of arrows $a_1 \ldots a_p$ in a quiver Q is called a *path of length* p if $s(a_i) = t(a_{i+1})$, this path is called a cycle if $s(a_p) = t(a_1)$. A path of length zero will be defined as a vertex. A quiver is strongly connected if for every couple of vertices (v_1, v_2) there exists a path p such that $s(p) = v_1$ and $t(p) = v_2$.

A *dimension vector* of a quiver is a map $\alpha : V \to \mathbb{N}$, the size of a dimension vector is defined as $|\alpha| := \sum_{v\in V} \alpha_v$. A couple (Q, α) consisting of a quiver and a dimension vector is called a *quiver setting* and for every vertex $v \in V$, α_v is refered to as the dimension of v. A setting is called *genuine* if none of the vertices has dimension 0. For every vertex $v \in V$ we also define the dimension vector

$$\epsilon_v : V \longrightarrow \mathbb{N} : w \mapsto \begin{cases} 0 & v \neq w, \\ 1 & v = w. \end{cases}$$

An *α-dimensional complex representation* W of Q assigns to each vertex v a linear space $\mathbb{C}^{\alpha_v}$ and to each arrow a a matrix

$$W_a \in \mathsf{Mat}_{\alpha_{t(a)} \times \alpha_{s(a)}}(\mathbb{C})$$

The space of all α-dimensional representations is denoted by $\mathsf{Rep}_\alpha Q$.

$$\mathsf{Rep}_\alpha Q := \bigoplus_{a \in A} \mathsf{Mat}_{\alpha_{t(a)} \times \alpha_{s(a)}}(\mathbb{C})$$

To the dimension vector α we can also assign a reductive group

$$\mathsf{GL}_\alpha := \bigoplus_{v \in V} \mathsf{GL}_{\alpha_\mathsf{v}}(\mathbb{C}).$$

An element of this group, g, has a natural action on $\mathsf{Rep}_\alpha Q$:

$$W := (W_a)_{a \in A},\ W^g := (g_{t(a)} W_a g_{s(a)}^{-1})_{a \in A}$$

Using this definitions we can reformulate the Luna-slice theorem for representation spaces [19]

Theorem 1.12. Suppose $\mathsf{Rep}_n A$ is a reduced variety and $\rho \in \mathsf{Rep}_n A$ is a smooth point corresponding to a semisimple representation with decomposition

$$S_1^{\oplus a_1} \oplus \cdots \oplus S_k^{\oplus a_k}.$$

We can identify N_ρ canonically with $\mathsf{Rep}_{\alpha_\rho} Q_\rho$ where Q_ρ is the *local quiver* of ρ. Q_ρ has k vertices corresponding to the set $\{S_i\}$ of simple factors of ρ and between S_i and S_j the number of arrows equals

$$\mathsf{Dim}_\mathbb{C} \mathsf{Ext}(\rho, \rho).$$

The dimension vector α_ρ is defined to be $(a_1, \ldots, a_k)$, where the a_i are the multiplicities of the simple components in ρ. The action of Stab_ρ on N_ρ corresponds to the normal action of $\mathsf{GL}_{\alpha_\rho}$ on $\mathsf{Rep}_{\alpha_\rho} Q_\rho$.

By consequence there exists an étale isomorphism φ between an open neighborhood of the point $0 \in \mathsf{iss}_{\alpha_\rho} Q_\rho$ and an open neighborhood of $[\rho] \in \mathsf{iss}_n A$ mapping 0 to $[\rho]$. Near the point $[\rho]$ $\mathsf{iss}_n A$ is analytically isomorphic to the quotient $0 \in \mathsf{iss}_{\alpha_\rho} Q_\rho$.

So in order to study representation spaces locally one can restrict in many cases to quiver representations. The objective of the rest of the thesis is now to look at the local properties of quiver representations.

Quiver representations can be seen as representations of an algebra, called the path algebra. If we take all the paths, including the one with zero length, as a basis we can form a complex vector space $\mathbb{C}Q$. On this space we can put a noncommutative product, by concatenating paths. By the concatenation of two paths $a_1 \dots a_p$ and $b_1 \dots b_q$ we mean

$$a_1 \dots a_p \cdot b_1 \dots b_q := \begin{cases} a_1 \dots a_p b_1 \dots b_q & s(a_p) = t(b_1) \\ 0 & s(a_p) \neq t(b_1) \end{cases}$$

For a vertex v and a path p we define vp as p if p ends in v and zero else. On the other hand pv is p if this path starts in v and zero else.

The vector space $\mathbb{C}Q$ equiped with this product is called the *path algebra*. The set of vertices forms a set of mutually orthogonal idempotents, for this algebra.

Suppose we have an n-dimensional representation ρ of the path algebra, then we can decompose the vector space $\mathbb{C}^n$ into a direct sum

$$\mathbb{C}_n := \rho(v_1)\mathbb{C}^n \oplus \dots \oplus \rho(v_k)\mathbb{C}^n.$$

Note that $\rho(v_i)$ acts like the identity on $\rho(v_i)\mathbb{C}^n$. Chosing bases in $\rho(v_i)\mathbb{C}^n$ we can associate with every arrow a of Q a matrix W_a corresponding to the map

$$\rho(a)|_{\rho(s(a))\mathbb{C}^n} : \rho(s(a))\mathbb{C}^n \to \rho(t(a))\mathbb{C}^n$$

These matrices give us a quiver representation W with dimension

$$\alpha : V \mapsto \mathbb{N} : v_i \mapsto \mathsf{Dim}\rho(v_i)\mathbb{C}^n.$$

The identification of ρ with a quiver representation depends on the choice of the base and hence there is still an action of GL_α working on this representation. Making the quotient we can say that every equivalence class of n-dimensional $\mathbb{C}Q$-representations defines uniquely an α-dimensional representation class of Q, for a certain α of size n. In symbols we have

$$\mathsf{iss}_n\mathbb{C}Q \cong \bigsqcup_{|\alpha|=n} \mathsf{iss}_\alpha Q.$$

Viewing quiver representations of a quiver as representations of a algebra we easily translate the concepts simple and semisimple representation to the quiver language.

A representation W is called *simple* if the only collections of subspaces $(\mathsf{V}_v)_{v\in V}$, $\mathsf{V}_v \subseteq \mathbb{C}^{\alpha_v}$ having the property

$$\forall a \in A : W_a \mathsf{V}_{s(a)} \subset \mathsf{V}_{t(a)}$$

are the trivial ones (i.e. the collection of zero-dimensional subspaces and $(\mathbb{C}^{\alpha_v})_{v\in V}$).

The direct sum $W \oplus W'$ of two representations W, W' has as dimension vector the sum of the two dimension vectors and as matrices $(W \oplus W')_a := W_a \oplus W'_a$. A representation equivalent to a direct sum of simple representations is called *semisimple*.

From this point of view an orbit of a quiver representation is closed if and only if this representation is semisimple. So one can also consider $\mathsf{iss}_\alpha Q$ as the space classifying all semisimple α-dimensional representation classes.

Schur's lemma also holds for quiver representations and it is trivial to genereralize the Luna slice theorem for the case of quiver representations.

In the case of quiver representations we can easily calculate the dimension of the extension space between two quivers.

Lemma 1.13. Suppose W_1 and W_2 are simple representations of Q with dimension vectors α_1 and α_2 then

$$\mathsf{Dim}_{\mathbb{C}}\mathsf{Ext}(W_1, W_2) = \delta_{W_1,W_2} - \chi_Q(\alpha_1, \alpha_2)$$

The proof of this theorem can be found in [13] along with additional information about quiver representations. Using this lemma we can reformulate the slice theorem for the case of quiver representations.

Theorem 1.14 (Le Bruyn-Procesi). For every point $p \in \mathsf{iss}_\alpha Q$, corresponding to a semisimple quiver representation

$$S_1^{\oplus a_1} \oplus \cdots \oplus S_k^{\oplus a_k},\ (S_i \text{ has dimension vector} \alpha_i)$$

we have an étale isomophism between an open neighborhood of the zero representation in $\mathsf{iss}_{\alpha_p} Q_p$ and an open neighborhood of p. Q_p has k vertices and between the i^{th} and the j^{th} vertex there are

$$\delta_{ij} - \chi_Q(\alpha_i, \alpha_j).$$

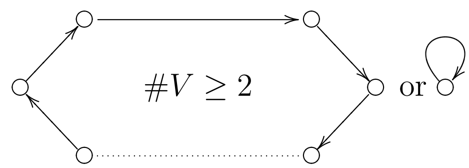
The dimension vector α_p assigns to the i^{th} vertex dimension a_i.

Note that the structure of the local quiver does not depend on the structure of the simple representations but only of its dimension vectors. So to apply this theorem we need a descripton of the dimension vectors of simple quiver representations.

We recall another result from Le Bruyn and Procesi [6].

Theorem 1.15. Let (Q, α) be a genuine quiver setting. There exist simple representations of dimension vector α if and only if

- If Q is of the form

  and $\alpha = \mathbf{1}$ (this is the constant map from the vertices to 1).

- Q is not of the form above, but strongly connected and

$$\forall v \in V : \chi_Q(\alpha, \epsilon_v) \leq 0 \text{ and } \chi_Q(\epsilon_v, \alpha) \leq 0$$

If (Q, α) is not genuine, there exist simple representations if and only if there exist for the genuine quiver setting obtained by deleting all vertices with dimension zero.

In the above cases the dimension of $\mathsf{iss}_\alpha Q$ is $1 - \chi_Q(\alpha, \alpha)$, when α is a simple dimension vector. If this dimension is zero then there is only one simple representation class with dimension vector α, this only occurs when the dimension vector is of the form ϵ_v where v is a vertex without loops.

Chapter 2

Coregular Quiver Settings

2.1 Generalities about the ring of invariants

In order to study $\mathsf{iss}_\alpha Q$ more closely, we recall some of the results of the article [6] by Le Bruyn and Procesi, which studies the local structure of the invariant ring $\mathbb{C}[\mathsf{iss}_\alpha Q]$.

As we know from the previous chapter a sequence of arrows $a_1 \dots a_p$ in a quiver Q is called a *path of length p* if $s(a_i) = t(a_{i+1})$, this path is called a cycle if $s(a_p) = t(a_1)$. To a cycle we can associate a polynomial function

$$f_c : \mathsf{Rep}_\alpha Q \to \mathbb{C} : W \mapsto \mathsf{Tr}(W_{a_1} \cdots W_{a_p})$$

which is definitely GL_α-invariant. Two cycles that are a cyclic permutation of each other give the same polynomial invariant, because of the basic properties of the trace map. Two such cycles are called equivalent.

A cycle $a_1 \dots a_p$ is called *primitive* if every arrow has a different starting vertex. This means that the cycle runs through each vertex at most 1 time. It is easy to see that every cycle has a decomposition in primitive cycles. It is however not true that the corresponding polynomial invariant decomposes to a product of the polynomial functions of the primitive cycles.

We will call a cycle *quasi-primitive* for a dimension vector α if the vertices that are ran through more than once have dimension bigger than 1. By cyclicly permuting a cycle and splitting the trace of a product of two 1×1 matrices into a product of traces, we can always decompose an f_c into a product of traces of quasi-primitive cycles. We now have the following result

Theorem 2.1 (Le Bruyn-Procesi). $\mathbb{C}[\mathsf{iss}_\alpha Q]$ is generated by all f_c where c is a quasi-primitive cycle of degree smaller than $|\alpha|^2+1$. We can turn $\mathbb{C}[\mathsf{iss}_\alpha Q]$ into a graded ring by giving f_c the length of its cycle as degree.

The proof of this theorem can be found in [6].

Definition 2.2. Define a partial ordering on the set of quivers as follows A quiver $Q' = (V', A', s', t')$ is smaller than $Q = (V, A, s, t)$ if (up to isomorphism)

$$V' \subseteq V,\ A' \subseteq A,\ s' = s|_{A'} \text{ and } t' = t|_{A'},$$

Q' is called a *subquiver* of Q.

Lemma 2.3. If $\mathsf{iss}_\alpha Q$ is smooth and $Q' \leq Q$ then $\mathsf{iss}_{\alpha'} Q'$ is also smooth, where $\alpha' := \alpha|_{V'}$

Proof. We have an embedding

$$\mathsf{Rep}_{\alpha'} Q' \hookrightarrow \mathsf{Rep}_\alpha Q$$

by assigning to the additional arrows in Q zero matrices. So

$$\mathbb{C}[\mathsf{Rep}_\alpha Q] \twoheadrightarrow \mathbb{C}[\mathsf{Rep}_{\alpha'} Q'] \ \Rightarrow\ \mathbb{C}[\mathsf{Rep}_\alpha Q]^{\mathsf{GL}_\alpha} \twoheadrightarrow \mathbb{C}[\mathsf{Rep}_{\alpha'} Q']^{\mathsf{GL}_\alpha}\ .$$

Because the action of GL_α on $\mathsf{Rep}_{\alpha'} Q'$ reduces to that of $\mathsf{GL}_{\alpha'}$, $\mathbb{C}[\mathsf{iss}_{\alpha'} Q']$ is a quotient ring of $\mathbb{C}[\mathsf{iss}_\alpha Q] = \mathbb{C}[X_1, \ldots, X_n]$. The only relations that we have to divide out are the X_i that correspond to a cycle containing one of the additional arrows we put zero, so $\mathbb{C}[\mathsf{iss}_{\alpha'} Q']$ is just a polynomial ring with less variables. $\square$

Two vertices v and w are said to be *strongly connected* if there is a path from v to w and vice versa. It is easy to check that this relation is an equivalence so we can divide the set of vertices into equivalence classes V_i. The subquiver Q_i having V_i as set of vertices, and as arrows all arrows between vertices of V_i is called a *strongly connected component* of Q.

Lemma 2.4.

1. If (Q, α) is a quiver setting then

   $$\mathbb{C}[\mathsf{iss}_\alpha Q] := \bigotimes_i \mathbb{C}[\mathsf{iss}_{\alpha_i} Q_i]$$

   where $Q_i = (V_i, A_i, s_i, t_i)$ are the strongly connected components of Q and $\alpha_i := \alpha|_{V_i}$.

2. $\mathsf{iss}_\alpha Q$ is smooth if and only if the $\mathsf{iss}_\alpha Q_i$ of all its strongly connected components are smooth.

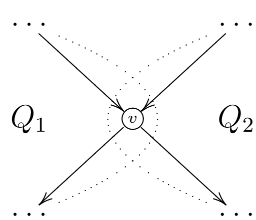
Proof.

1. By theorem 2.1 $\mathbb{C}[\mathsf{iss}_\alpha Q]$ is generated by the traces of cycles. Every cycle belongs to a certain connected component of Q. Between f_c's coming from cycles of different components there cannot be any relations, so we can consider the ring of invariants as a tensor-products of the rings of invariants different strongly connected components.

2. If all the strongly connected components are coregular the ring of invariants of the total quiver setting will be the tensor product of polynomial rings and hence a polynomial ring. The inverse implication follows directly from lemma 2.3.

∎

Definition 2.5. A quiver $Q = (V, A, s, t)$ is said to be the *connected sum* of 2 subquivers $Q_1 = (V_1, A_1, s_1, t_1)$ and $Q_1 = (V_2, A_2, s_2, t_2)$ at the vertex v, if the two subquivers make up the whole quiver and only intersect in the vertex v. So in symbols $V = V_1 \cup V_2$, $A = A_1 \cup A_2$, $V_1 \cap V_2 = \{v\}$ and $A_1 \cap A_2 = \emptyset$.

$Q_1 \#_v Q_2 :=$ 

If we connect three or more components we write $Q_1 \#_v Q_2 \#_w Q_3$ instead of $(Q_1 \#_v Q_2) \#_w Q_3$ for sake of simplicity.

Lemma 2.6. Suppose $Q = Q_1 \#_v Q_2$ and $\alpha_v = 1$ then

$$\mathbb{C}[\mathsf{iss}_\alpha Q] := \mathbb{C}[\mathsf{iss}_{\alpha_1} Q_1] \otimes \mathbb{C}[\mathsf{iss}_{\alpha_2} Q_2]$$

where $\alpha_i := \alpha|_{Q_i}$.

Proof. By theorem 2.1 $\mathbb{C}[\mathsf{iss}_\alpha Q]$ is generated by the traces of quasi-primitive cycles. Because the dimension of v is one every quasi-primitive cycle is either in subquiver Q_1 or Q_2 and there cannot be any relations between invariants coming from cycles in different subquivers. This implies that the ring of invariants of (Q, α) is the tensorproduct of the rings of invariants of the two subquiver settings. ∎

Finally we can restrict to quivers without loops. This is a consequence of the first fundamental theorem of GL_n, see for example [18, II.4.1 p116].

Theorem 2.7 (first fundamental theorem of GL_n). For every $l, n, m \in \mathbb{N}$, the affine quotient

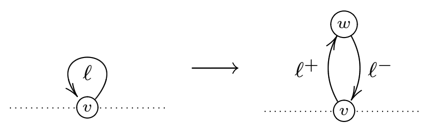
$$\mathsf{Mat}_{l\times n}(\mathbb{C}) \oplus \mathsf{Mat}_{n\times m}(\mathbb{C})/\mathsf{GL}_n(\mathbb{C})$$

is isomorphic to the space of all $l \times m$-matrices of rank smaller than n. The identification is obtained via the projection map

$$\pi : \mathsf{Mat}_{l\times n}(\mathbb{C}) \oplus \mathsf{Mat}_{n\times m}(\mathbb{C}) \to \mathsf{Mat}_{l\times m}(\mathbb{C}) : (A, B) \mapsto AB$$

In particular when $n \geq l, m$ π is surjective and $\mathsf{Mat}_{l\times n}(\mathbb{C})\oplus\mathsf{Mat}_{n\times m}(\mathbb{C})/\mathsf{GL}_n(\mathbb{C}) \cong \mathsf{Mat}_{l\times m}(\mathbb{C})$

If we have a quiver setting with loops than we can construct a new quiver setting $(Q^\times, \alpha^\times)$ such that $\mathsf{iss}_{\alpha^\times} Q^\times$ is isomorphic to the original $\mathsf{iss}_\alpha Q$. We alter every loop in the original quiver into a vertex and two arrows as in the picture.



The dimension at w is bigger or equal than on the vertex v ($\alpha^\times_w \geq \alpha_v$). If we divide out the base change action of on the vertex w, the quotient space $\mathsf{Rep}_{\alpha^\times} Q^\times/\mathsf{GL}_{\alpha^\times_w}(\mathbb{C})$ is isomorphic to $\mathsf{Rep}_\alpha Q$ by the fundamental theorem.

Lemma 2.8.

$$\mathsf{iss}_{\alpha^\times} Q^\times \cong (\mathsf{Rep}_{\alpha^\times} Q^\times/\mathsf{GL}_{\alpha^\times_w}(\mathbb{C}))/\mathsf{GL}_\alpha \cong \mathsf{iss}_\alpha Q$$

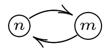
2.2 The Symmetric Case

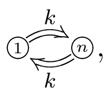
The main theorem we prove in this section is a classification of all coregular symmetric quiver settings and can be found in [3]. Lemmas 2.4 and 2.8 allow us to consider only strongly connected quivers without any loops. The result we will obtain is:

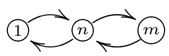
Theorem. A symmetric quiver setting without loops (Q, α) is coregular if and only if the following conditions are satisfied:

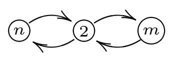
- Q is treelike, by which we mean that the underlying graph, having the same vertices as Q and 1 edge between two vertices whenever there is at least one arrow between them in Q, is a tree.
- The branching vertices (i.e. vertices having more than two incoming arrows) have dimension 1.

- The quiver setting is constructed by sticking together subquiver settings of the types shown below identifying only vertices with dimension 1.

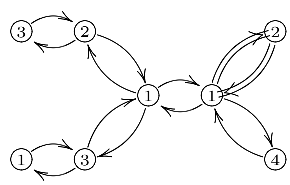
  I 

  II , $k \leq n$

  III 

  IV ,

As an example illustrating this theorem we show a coregular quiver setting made by sticking together 2 settings of type I, and 1 of type II, III, and IV.

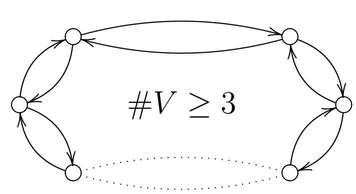


The proof of the theorem is based on two observations.

1. If a quiver setting is coregular then all its subquiver settings and all its possible local quiver settings are coregular.

2. If one sticks together two quiver settings by identifying a vertex with dimension 1, the ring of invariants of this new setting will be the tensor product of the rings of invariants of the two original quiver settings.

First one proves that a coregular quiver setting must be treelike because of observation 1 and the fact that a quiver of the form



is not coregular for any dimension vector. A similar argument is used to prove that the branching vertices must have dimension 1.

Observation 2 allows us now to cut the tree into pieces, (or in the algebraic way decomposing the ring of invariants into a tensor product) and to look at the pieces separately. Finally one concludes the proof by a classification of all coregular pieces that cannot be reduced into smaller ones.

2.2.1 The necessary conditions

First we determine some neccesary conditions for a symmetric quiver setting to be coregular in the next section we will use these conditions to generate all coregular quiver settings.

We first look at a simple case and then we rule out more complex quiver settings by looking at the local quiver of some decompositions. In the following pictures of quiver settings, we will write down the dimensions inside the corresponding vertices.

Lemma 2.9. The following quiver setting is not coregular if $k > 1$.

$$\begin{tikzcd} 1 \arrow[r, bend left, "k"] & 1 \arrow[l, bend left, "k"] \end{tikzcd} \quad (*)$$

Proof. The representation space is spanned by all the cycles

$$X_{ij} = f_{a_i b_j}$$

where a_i stands for one of the arrows to the right and b_j one to the left. All these cycles are neccesary to generate the algebra, because the representation for which all the arrows are zero except a_i and b_j is not equivalent to the zero and has as values in the cycles all zero's except for X_{ij}. The relations between the cycles are of the form

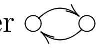
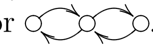
$$X_{ij}X_{kl} = X_{il}X_{kj}$$

These relations prevent $\mathsf{iss}_\alpha Q$ from being an affine space. The only way to make $\mathsf{iss}_\alpha Q$ into an affine space is that there is only 1 such cycle. □

We will use this lemma in the following way. Suppose that (Q, α) is a quiver setting and that β and γ are dimension vectors of simples for Q (this can be checked by 1.15) such that $\beta + \gamma \leq \alpha$.

If both β and γ are not of the form ϵ_v for some vertex v, we can construct a representation that is the direct sum of representations If (Q, α) is not genuine, there exist simple representations if and only if there exist for the genuine quiver setting obtained by deleting all vertices with dimension zero.

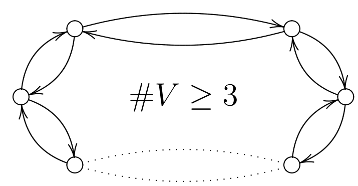
Suppose we have a quiver setting which has a decomposition that contains two simple factors that occur with multiplicity 1. The corresponding local quiver will contain a subsetting of the form $(*)$ with $k = -\chi_Q(\beta, \gamma)$ where β and γ are the dimension vectors of the simple factors. To determine whether β and γ are dimension vectors of simples we can use 1.15.

Lemma 2.10. Suppose that (Q, α) is a coregular symmetric strongly connected quiver without loops and $\forall v \in V : \alpha(v) > 1$ then Q is either  or .

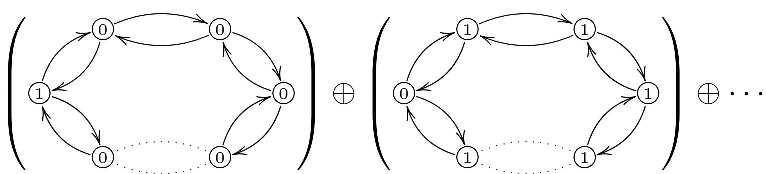
Proof. We can make a decomposition containing 2 *different* simple components with dimension vector $\mathbf{1}$ and all the other components with dimension vectors of the form ϵ_v.

The number of arrows in the local quiver between the first two components is $\chi(\mathbf{1}, \mathbf{1}) = \#A - \#V$. In order to be coregular this number must be at most 1. Because the quiver is symmetric and strongly connected $\#A \geq 2(\#V - 1)$ so $1 \geq \chi(\mathbf{1}, \mathbf{1}) \geq \#V - 2$. The only quivers satisfying $\#V \leq 3$ and $\#A \leq \#V + 1$ are the one listed above. $\square$

Lemma 2.11. The following quiver is not coregular for any dimension vector



Proof. By the previous lemma we can suppose that the dimension of the left vertex is 1. Make the folowing decomposition:

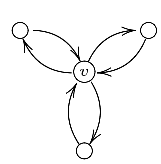


Where the dots stand for components with dimension vectors of the form ϵ_v. The number of arrows in the local quiver between the first two components is 2 so (Q, α) is not coregular. $\square$

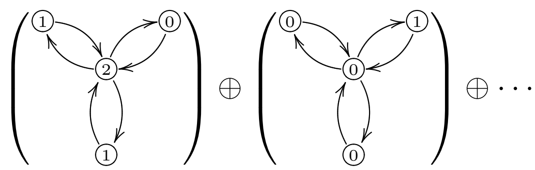
The lemma above, in combination with lemma 2.3 shows us that if we look at the underlying graph of a coregular quiver setting (Q, α) (having the same vertices as Q, and 1 edge between two vertices if there is an arrow between them), this graph must have the form of a tree. So we have restrictions on the form of the quiver.

There are also restrictions on the possible dimension vectors. We will determine at which vertices the dimension vector has to be 1.

Lemma 2.12. The following quiver setting is not coregular if the dimension in the centre v is bigger than 1.



Proof. If the dimension vector of the centre is bigger than 1 then by 2.10 we can suppose that at least 1 of the dimensions of the other vertices is 1 (take this to be the upper right one). We can find a decomposition of the form

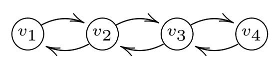


The number of arrows between the first and the second simple component equals

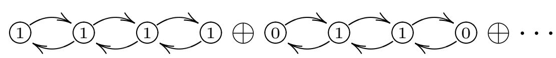
$$-\begin{pmatrix}2 & 1 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & -1 & -1 & -1\\ -1 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ -1 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = 2$$

$\square$

Lemma 2.13. The following quiver setting is not coregular if $v_2, v_3 \geq 2$



Proof. We have a decomposition



The number of arrows between the first and the second simple component is again given by

$$-\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1 & 0 & 0\\ -1 & 1 & -1 & 0\\ 0 & -1 & 1 & -1\\ 0 & 0 & -1 & 1\end{pmatrix}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} = 2$$

$\square$

Theorem 2.14. If a quiver setting (Q, α) (Q is symmetric and connected without loops) is coregular then Q is a connected sum

$$Q := Q_1 \#_{v_1} Q_2 \#_{v_2} \cdots \#_{v_{k-1}} Q_k,$$

where the Q_i have at most 3 vertices. and $\alpha_{v_j} = 1,\ j = 1, \dots, k-1$.

Proof. If a symmetric connected quiver setting is coregular and it has more than 3 vertices then it is treelike. Cutting at the vertices with dimension 1, we can consider Q as a connected sum of smaller components. Because branching vertices have dimension 1 and there are no 2 consecutive vertices with dimension bigger than 1 unless they are at the end of a branch, the components of this connected sum have at most 3 vertices. $\square$

2.2.2 The characterisation result

In this section we determine all symmetric coregular quiver settings with 2 or 3 vertices. Afterwards we combine them to obtain bigger quivers which give us all symmetric quiver settings that are coregular.

Lemma 2.15. The quiver setting

$$\begin{tikzcd} n \arrow[r, bend left, "k"] & m \arrow[l, bend left, "k"] \end{tikzcd}, \quad n \le m$$

is coregular if and only if $k = 1$ or $1 = n \le k \le m$.

Proof. if $k = 1$ then lemma 2.8 shows that the space is equal to that of the quiver with one vertex and one loop, such that the dimension vector is n. This problem is the same as the conjugacy problem of matrices, which is known to have a smooth space (see for example [18]).

If $k > 1$ then by lemma 2.10 at least n must be 1. If $k > m$ then we can make the following decomposition in simples:

$$\begin{tikzcd} 1 \arrow[r, bend left, "k"] & m-1 \arrow[l, bend left, "k"] \end{tikzcd} \oplus \begin{tikzcd} 0 \arrow[r, bend left, "k"] & 1 \arrow[l, bend left, "k"] \end{tikzcd}$$

Computing the number of arrows in the local quiver gives us

$$-\begin{pmatrix} 1 & m-1 \end{pmatrix} \begin{pmatrix} 1 & -k \\ -k & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = k - m + 1 > 1.$$

If $k = m$ then there are exactly k^2 quasi-primitive cycles. Moreover (Q, α) is simple and the dimension of $\mathsf{iss}_\alpha Q$ is $\chi_Q(\alpha, \alpha) = k^2$. Hence there can be no relations between the generators of $\mathbb{C}[\mathsf{iss}_\alpha Q]$. By lemma 2.3 is the case $k < m$ also coregular. ☐

For quivers with 3 vertices we only have to look at the settings where the dimension of the middle vertex is bigger than 1 because otherwise we can consider it as a connected sum of two quiver settings with 2 vertices.

Lemma 2.16. The following quiver setting is not coregular if $v_2 \ge 3$ and $v_1, v_3 \ge 2$

$$\begin{tikzcd} v_1 \arrow[r, bend left] & v_2 \arrow[l, bend left] \arrow[r, bend left] & v_3 \arrow[l, bend left] \end{tikzcd}$$

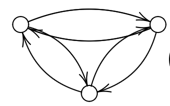
Proof. We have a decomposition

$$\begin{tikzcd} 1 \arrow[r, bend left] & 1 \arrow[l, bend left] \arrow[r, bend left] & 1 \arrow[l, bend left] \end{tikzcd} \oplus \begin{tikzcd} 0 \arrow[r, bend left] & 1 \arrow[l, bend left] \arrow[r, bend left] & 1 \arrow[l, bend left] \end{tikzcd} \oplus \begin{tikzcd} 1 \arrow[r, bend left] & 1 \arrow[l, bend left] \arrow[r, bend left] & 0 \arrow[l, bend left] \end{tikzcd} \oplus \cdots$$

computing the arrows:

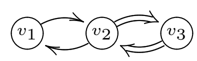
$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = -1$$

shows that the associated local quiver has a subquiver of the form

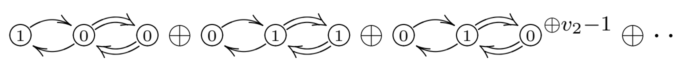

(++)

which is not coregular according to lemma 2.11. □

Lemma 2.17. The following quiver setting is not coregular if $v_2 \geq 2$

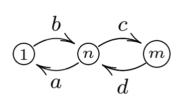


Proof. We have a decomposition



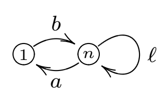
computing the arrows as in the previous lemma shows that the associated local quiver has also a subquiver of the form (++). □

Lemma 2.18. The quiver setting



is coregular.

Proof. If $m > n$ the situation is the same as



this quiver setting has simple representations by 1.15 and the dimension of quotient space $\mathsf{iss}_\alpha Q$ is

$$1 - \begin{pmatrix} 1 & n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ n \end{pmatrix} = 2n$$

The only quasi-primitive cycles are of the form

$$\ell^i \text{ and } a\ell^i b$$

By the Cayley-Hamilton identity for V_ℓ, we only need the $2n$ quasi-primitive cycles $\ell^i, 1 \leq i \leq n$ and $a\ell^i b, 0 \leq i \leq n-1$ to generate the ring of invariants. Because the dimension of $\mathsf{iss}_\alpha Q$ is $2n$, $\mathbb{C}[\mathsf{iss}_\alpha Q]$ must be polynomial.

If $m < n$ a general representation consist of 2 big matrices as shown in the picture

$$\begin{tikzcd} 1 \arrow[r, bend left, "V_b"] & n \arrow[l, bend left, "V_a"] \arrow[r, bend left, "V_c"] & m \arrow[l, bend left, "V_d"] \end{tikzcd} \quad \Rightarrow \quad \mathbb{C}^1 \oplus \mathbb{C}^m \xrightarrow{V_b \oplus V_d} \mathbb{C}^n \xrightarrow{V_a \oplus V_c} \mathbb{C}^1 \oplus \mathbb{C}^m$$

If we divide out the action of base change on $\mathbb{C}^n$ using the fundamental theorem 2.7, the quotient gives us one big composite map which corresponds to a new quiver situation

$$\mathbb{C}^1 \oplus \mathbb{C}^m \xrightarrow{(V_a \oplus V_c)(V_b \oplus V_d)} \mathbb{C}^1 \oplus \mathbb{C}^m \quad \Rightarrow \begin{tikzcd} 1 \arrow[loop left] \arrow[r, bend left] & m \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$$

This new quiver setting is coregular because it is the connected sum of two coregular quiver settings: one vertex of dimension 1 with one loop and the quiver setting we encountered in the first part of the proof. $\square$

Lemma 2.19. The quiver setting

$$\begin{tikzcd} n \arrow[r, bend left] & 2 \arrow[l, bend left] \arrow[r, bend left] & m \arrow[l, bend left] \end{tikzcd}$$

is coregular.

Proof. If both n and m are bigger then 1, this problem is the same as a vertex with two loops, or two simultaneously conjugated 2×2-matrices. As we know from [22], this problem is coregular and its ring of invariants is generated by

$$\mathsf{Tr}A, \mathsf{Tr}B, \mathsf{Tr}A^2, \mathsf{Tr}B^2 \text{ and } \mathsf{Tr}AB,$$

where A and B are the matrices corresponding to the two loops. If $n = 1$, we can take the same invariants as above, taking for A only rank 1 matrices, therefore $\mathsf{Tr}A^2 = (\mathsf{Tr}A)^2$ is the only extra relation. This means that the ring of invariants is indeed a polynomial ring generated by

$$\mathsf{Tr}A, \mathsf{Tr}B, \mathsf{Tr}B^2 \text{ and } \mathsf{Tr}AB.$$

If both n and m are 1 we are in the situation of the previous lemma. $\square$

Keeping 2.16 and 2.17 in mind, the last two lemmas give us all coregular quiver settings with 3 vertices that are not the connected sum of smaller ones.

Combining all the results we obtain the classification of all symmetric coregular quiver settings.

Theorem 2.20. A quiver setting (Q, α) (Q is symmetric and connected without loops) is coregular if and only if Q is a connected sum

$$Q := Q_1 \underset{v_1}{\#} Q_2 \underset{v_2}{\#} \cdots \underset{v_{k-1}}{\#} Q_k,$$

where the (Q_i, α_i) are of the form

- $n \rightleftarrows m$
- $1 \overset{k}{\underset{k}{\rightleftarrows}} n$, $k \leq n$
- $1 \rightleftarrows n \rightleftarrows m$
- $n \rightleftarrows 2 \rightleftarrows m$

and $\alpha_{v_j} = 1,\ j = 1, \ldots, k-1$

Proof. The proof follows from theorem 2.14 and the fact that the above list characterizes all coregular quiver settings with 3 or less vertices that cannot be written as a connected sum of smaller ones. □

2.3 The General Case

The technique we used for the symmetric cases is cannot be applied to the general case because it is impossible to make an exhaustive list of coregular quivers that cannot be split into connected sums. Therefore we need more powerfull methods for the reduction of quiver settings to simpler settings.

The outline for the classification of the general case starts with the introduction of 3 methods to reduce quiver settings. Then we prove that the quiver settings that are coregular and cannot be reduced by these reduction steps contain only one vertex. This result appeared in [4].

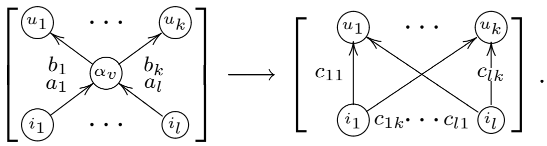
2.3.1 Reduction Steps

As we stated above we want to apply some kind of reduction on quivers. By this we mean that if we start from a general quiver setting (Q, α), we want to construct a new quiver setting with less vertices or arrows but has the same (or a closely related ring of invariants). In this section we will consider 3 different kinds of reductions.

Lemma 2.21 (Reduction $\mathcal{R}_I$: Removing Vertices). Suppose (Q, α) is a quiver setting and v is a vertex without loops such that

$$\chi_Q(\alpha, \epsilon_v) \geq 0 \text{ or } \chi_Q(\epsilon_v, \alpha) \geq 0.$$

Construct a new quiver setting (Q', α') by changing Q:



(Some of the vertices in the picture may be the same). Those two quiver settings have isomorphic rings of invariants.

Proof. We can split up the representation space into the following direct sum

$$\begin{aligned}\mathsf{Rep}_\alpha Q &= \underbrace{\bigoplus_{a,\ s(a)=v} \mathsf{Mat}_{\alpha_{t(a)}\times\alpha_{s(a)}}(\mathbb{C})}_{\text{arrows starting in } v} \oplus \underbrace{\bigoplus_{a,\ t(a)=v} \mathsf{Mat}_{\alpha_{t(a)}\times\alpha_{s(a)}}(\mathbb{C})}_{\text{arrows terminating in } v} \oplus \text{Rest} \\ &= \mathsf{Mat}_{\sum_{s(a)=v}\alpha_{t(a)}\times\alpha_v}(\mathbb{C}) \oplus \mathsf{Mat}_{\alpha_v\times\sum_{t(a)=v}\alpha_{s(a)}}(\mathbb{C}) \oplus \text{ Rest} \\ &= \mathsf{Mat}_{\alpha_v-\chi(\alpha,\epsilon_v)\times\alpha_v}(\mathbb{C}) \oplus \mathsf{Mat}_{\alpha_v\times\alpha_v-\chi(\epsilon_v,\alpha)}(\mathbb{C}) \oplus \text{ Rest}\end{aligned}$$

The $\mathsf{GL}_{\alpha_v}(\mathbb{C})$-part only acts on the first two terms and not on the rest term. So if we take the quotient corresponding to $\mathsf{GL}_{\alpha_v}(\mathbb{C})$ we only have to consider the first two terms.

By the fist fundamental theorem 2.7 and keeping in mind that either $\chi_Q(\alpha, \epsilon_v) \geq 0$ or $\chi_Q(\epsilon_v, \alpha) \geq 0$ the quotient space is equal to

$$\mathsf{Mat}_{\alpha_v-\chi(\alpha,\epsilon_v)\times\alpha_v-\chi(\epsilon_v,\alpha)}(\mathbb{C}) \oplus \text{ Rest.}$$

This space can be decomposed in the following way:

$$\bigoplus_{a,\ t(a)=vb,\ s(b)=v} \mathsf{Mat}_{\alpha_{t(b)}\times\alpha_{s(a)}}(\mathbb{C}) \oplus \text{ Rest.}$$

This direct sum is the same as the representation space of the new quiver setting (Q', α'). ☐

Lemma 2.22 (Reduction $\mathcal{R}_{II}$: Removing loops of dimension 1). Suppose that (Q, α) is a quiver setting and v a vertex with k loops and $\alpha_v = 1$. Take Q' the corresponding quiver without loops, then the following identity hold

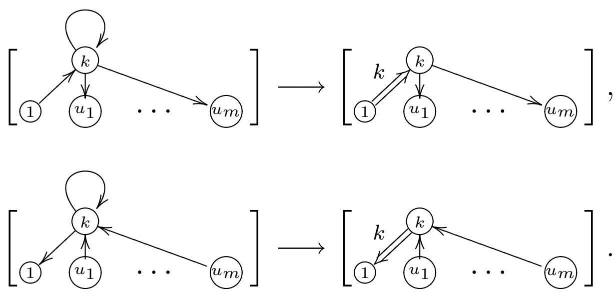
$$\mathbb{C}[\mathsf{iss}_\alpha Q] \cong \mathbb{C}[\mathsf{iss}_\alpha Q'] \otimes \mathbb{C}[X_1, \cdots, X_k]$$

Proof. This follows easily from the fact that the GL_α-action on such a loop is trivial. ☐

Lemma 2.23 (Reduction $\mathcal{R}_{III}$: Removing a loop of higher dimension). Suppose (Q, α) is a quiver setting and v is a vertex of dimension $k \geq 2$ with one loop such that

$$\chi_Q(\alpha, \epsilon_v) = -1 \text{ or } \chi_Q(\epsilon_v, \alpha) = -1.$$

Construct a new quiver setting (Q', α') by changing (Q, α):



We have the following identity:

$$\mathbb{C}[\mathsf{iss}_\alpha Q] \cong \mathbb{C}[\mathsf{iss}_{\alpha'} Q'] \otimes \mathbb{C}[X_1, \ldots, X_k]$$

Proof. We only prove this for the first case. Call the loop in the first quiver ℓ and the incoming arrow a. Call the incoming arrows in the second quiver $c_i, i = 0, \ldots, k-1$.

There is a map

$$\pi : \mathsf{Rep}_\alpha Q \to \mathsf{Rep}_{\alpha'} Q' \times \mathbb{C}^k : V \mapsto (V', \mathsf{Tr} V_\ell, \ldots, \mathsf{Tr} V_{\ell^k}) \text{ with } V'_{c_i} := V_\ell^i V_a$$

Suppose $(V', x_1, \ldots, x_k) \in \mathsf{Rep}_{\alpha'} Q' \times \mathbb{C}^k \in$ such that $(x_1, \ldots, x_k)$ correspond to the traces of powers of an invertible diagonal matrix D with k different eigenvalues $(\lambda_i, i = 1, \ldots, k)$ and the matrix A made of the columns $(V_{c_i}, i = 0, \ldots, k-1)$ is invertible. The image of representation

$$V \in \mathsf{Rep}_\alpha Q : V_a = V'_{c_0}, V_\ell = A \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix}^{-1} D \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix} A^{-1}$$

under π is $(V', x_1, \ldots, x_k)$ because

$$\begin{aligned} V_\ell^i V_a &= A \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix}^{-1} D^i \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix} A^{-1} V'_{c_0} \\ &= A \begin{pmatrix} \lambda_1^0 & \cdots & \lambda_1^{k-1} \\ \vdots & & \vdots \\ \lambda_k^0 & \cdots & \lambda_k^{k-1} \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1^i \\ \vdots \\ \lambda_k^i \end{pmatrix} \\ &= V_{c_i} \end{aligned}$$

and the traces of V_ℓ are the same as the ones of D. The conditions we imposed on $(V', x_1, \ldots, x_k)$, implie that the image of π, U, is dense, and hence π is a dominant map.

We have a bijection between the generators of $\mathbb{C}[\mathsf{iss}_\alpha Q]$ and $\mathbb{C}[\mathsf{iss}_{\alpha'} Q'] \otimes \mathbb{C}[X_1, \ldots, X_k]$ by identifying

$$f_{\ell^i} \mapsto X_i, i = 1, \ldots, k \ , f_{\cdots a\ell^i \cdots} \mapsto f_{\cdots c_i \cdots}, i = 0, \ldots, k-1$$

Notice that higher orders of ℓ don't occur because of the Caley Hamilton identity on V_ℓ. So if n is the number of generators of $\mathbb{C}[\mathsf{iss}_\alpha Q]$, we have two maps

$$\begin{aligned} \phi &: \mathbb{C}[Y_1, \cdots Y_n] \to \mathbb{C}[\mathsf{iss}_\alpha Q] \subset \mathbb{C}[\mathsf{Rep}_\alpha Q], \\ \phi' &: \mathbb{C}[Y_1, \cdots Y_n] \to \mathbb{C}[\mathsf{iss}_{\alpha'} Q'] \otimes \mathbb{C}[X_1, \ldots, X_k] \subset \mathbb{C}[\mathsf{Rep}_{\alpha'} Q' \times \mathbb{C}^k]. \end{aligned}$$

Notice that we have that $\phi'(f) \circ \pi \equiv \phi(f)$ and $\phi(f) \circ \pi^{-1}|_U \equiv \phi'(f)|_U$. So if $\phi(f) = 0$ then also $\phi'(f)|_U = 0$. Because U is Zariski-open and dense in $\mathsf{Rep}_{\alpha'} Q' \times \mathbb{C}^2$, $\phi'(f) \equiv 0$. A similar argument holds for the inverse implication so $\mathsf{Ker}\phi = \mathsf{Ker}\phi'$. $\square$

We have now seen three possible reductions of a quiver setting which keep the ring of invariants intact or split of a tensor product with a polynomial ring. We can also apply the inverse steps of the reduction to add new vertices or loop while keeping the ring of invariants the same or tensoring it up with a polynomial ring. These inverse steps will be denoted as $\mathcal{R}_{\ldots}^{-1}$.

The previous three lemma's can now be summarized as

Theorem 2.24. Suppose that (Q, α) and (Q', α') are two quiver settings that can be transformed into each other using consecutive steps of the form $\mathcal{R}_I$, $\mathcal{R}_I^{-1}$, $\mathcal{R}_{II}$, $\mathcal{R}_{II}^{-1}$, $\mathcal{R}_{III}$ or $\mathcal{R}_{III}^{-1}$. Then (Q, α) is coregular if and only if (Q', α') is coregular.

Definition 2.25. A quiver setting (Q, α) such that none of the reduction steps $\mathcal{R}_I$, $\mathcal{R}_{II}$ or $\mathcal{R}_{III}$ can be applied, will be called *reduced*.

It remains to search for the reduced coregular quiver settings. We will prove that there are only a very limited number of them.

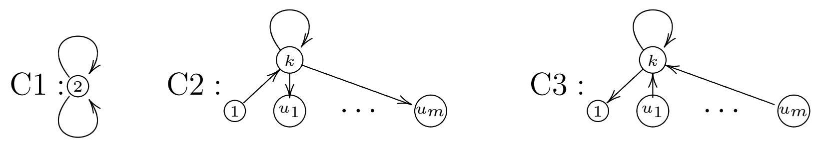
2.3.2 Reduced coregular quiver settings

First, we look at the case of loops.

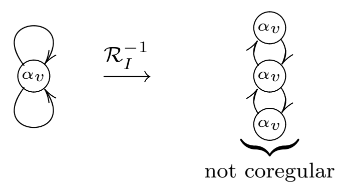
Lemma 2.26. Suppose (Q, α) is a coregular strongly connected quiver setting such that

$$\forall w \in V : \chi_Q(\alpha, \epsilon_w) < 0 \text{ and } \chi_Q(\epsilon_w, \alpha) < 0.$$

If v is a vertex with loops then $\alpha_v = 1$ or the neighborhood of v has the following form

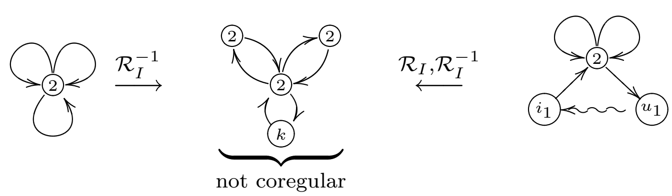


Proof. $\boxed{\text{1. if } \alpha_v \geq 3 \text{ there is only one loop in } v}$
Suppose that $\alpha_v \geq 3$ and there are at least two loops in v. In this case we have a subquiver as shown below. This subquiver can be transformed into a the symmetric quiver without loops using lemma 2.21. By 2.20 this symmetric setting is not coregular, if $\alpha_v > 2$



$\boxed{\text{2. If } \alpha_v = 2 \text{ we are in C1 or there is only 1 loop in } v}$

If $\alpha_v = 2$ and Q has either at least 3 loops or either two loops and a cyclic path through v, we can take again the corresponding subquivers and change them to a symmetric quiver without loops which is not coregular according to 2.20.



So the only possibility with more than one loop is C1.

$\boxed{\text{3. If } \alpha_v \geq 2 \text{ and there is only 1 loop in } v \text{ then we are in C2 or C3}}$

Suppose that the dimension in v is bigger than 1 and that there is only 1 loop. Consider the representation

$$W \oplus L \oplus \left(\bigoplus_{w \in V} S_w^{\oplus \alpha_w - 1 - \delta_{vw}} \right)$$

Where W is a simple representation with dimension vector $\mathbf{1}$, which is the constant map assigning 1 to every vertex. Such a representation exists by 1.15 because Q is strongly connected and the assumptions on the Euler matrix. S_w is the representation with dimension vector $\epsilon_w : v \mapsto \delta_{vw}$ and assigning to every arrow a zero matrix, while L is a representation with dimension vector ϵ_v which assigns to the loop in v a non-zero matrix.

For every vertex $w \neq v$ with dimension bigger than 1 the local quiver contains exactly one vertex corresponding to the simple representation S_w. For v there is at least one vertex in the local quiver coming from L, which has dimension 1. If $\alpha_v > 2$ there is an extra vertex from the S_v but we will not consider it because it does not change the proof.

The subquiver containing the vertices from L and $S_w, w \neq v$ is the same as in the original quiver because

$$\chi_Q(\epsilon_u, \epsilon_w) = \delta_{uw} - \#\{a | \textcircled{u} \xleftarrow{a} \textcircled{w}\}$$

In the local quiver we will draw the additional vertex coming from W as a square. The number of arrows from another vertex coming from S_w to the vertex coming from W is equal to $-\chi_Q(\mathbf{1}, \epsilon_w)$ and hence one less than the number of arrows leaving w in the original quiver. The same holds for the number of arrows in the opposite direction and for the arrows between L and W.

We will now look closely at the neighborhood of v.

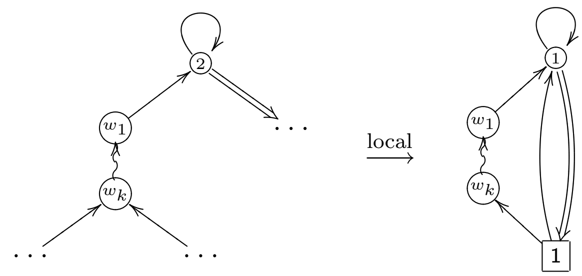
- $\boxed{\chi_Q(\epsilon_v, \mathbf{1}) \leq -2 \text{ and } \chi_Q(\mathbf{1}, \epsilon_v) \leq -2 \text{ is impossible}}$

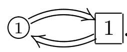
  The local quiver has a subquiver containing $\textcircled{1} \overset{\rightrightarrows}{\leftleftarrows} \boxed{1}$, and (Q, α) is not coregular. For (Q, α) to be a coregular quiver setting, one can suppose that either $\chi_Q(\epsilon_v, \mathbf{1}) = -1$ or $\chi_Q(\mathbf{1}, \epsilon_v) = -1$.

- $\boxed{\chi_Q(\epsilon_v, \mathbf{1}) = -1 \text{ and } \chi_Q(\mathbf{1}, \epsilon_v) \leq -2 \text{ implies C2.}}$

  Suppose w_1 is the unique vertex in Q such that $\chi_Q(\epsilon_v, \epsilon_{w_1}) = -1$ then $\alpha_{w_1} = 1$.

  If this were not the case there is a vertex correponding to S_{w_1} in the local quiver. If $\chi_Q(\mathbf{1}, \epsilon_{w_1}) = 0$ then the dimension of the unique vertex w_2 with an arrow to w_1 has strictly bigger dimension than w_1, otherwise $\chi_Q(\alpha, \epsilon_{w_1}) \geq 0$. The vertex w_2 corresponds again to a vertex in the local quiver. If $\chi_Q(\mathbf{1}, \epsilon_{w_2}) = 0$, the unique vertex w_3 with an arrow to w_2 has strictly bigger dimension than w_2. Proceeding this way one can find a sequence of vertices with increasing dimension, which attains a maximum in vertex w_k. Therefore $\chi_Q(\mathbf{1}, \epsilon_{w_k}) \leq -1$. This last vertex is in the local quiver connected with W, so one has a path from $\mathbf{1}$ to ϵ_v.



The local subquiver consisting of the vertices corresponding to W, S_v and the S_{w_i} is reducible via $\mathcal{R}_I$ to . So if $\alpha_{w_1} > 1$, (Q, α) is not coregular.

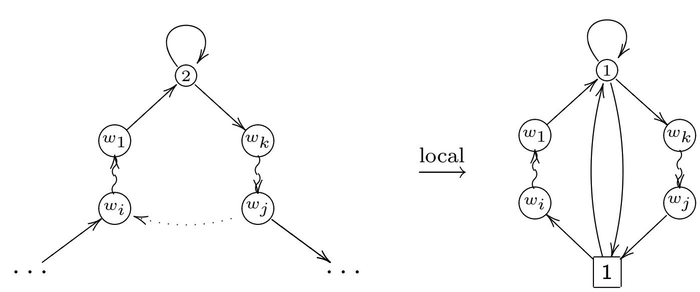
- $\boxed{\chi_Q(\epsilon_v, \mathbf{1}) \leq -2 \text{ and } \chi_Q(\mathbf{1}, \epsilon_v) = -1 \text{ implies C3.}}$

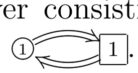
  Suppose w_1 is the unique vertex in Q such that $\chi_Q(\epsilon_{w_1}, \epsilon_v) = -1$ then $\alpha_{w_1} = 1$ because of similar reasons as above.

- $\boxed{\chi_Q(\epsilon_v, \mathbf{1}) = -1 \text{ and } \chi_Q(\epsilon_v, \mathbf{1}) = -1 \text{ implies C2 or C3.}}$

  Suppose w_1 is the unique vertex in Q such that $\chi_Q(\epsilon_v, \epsilon_{w_1}) = -1$ and w_k is the unique vertex in Q such that $\chi_Q(\epsilon_{w_k}, \epsilon_v) = -1$, then either $\alpha_{w_1} = 1$ or $\alpha_{w_k} = 1$.

  If this were not the case, consider the path connecting w_k and w_1 and call the intermediate vertices w_i, $1 < i < k$. Starting from w_1 we go back the path until α_{w_i} reaches a maximum. at that point we know that $\chi_Q(\mathbf{1}, \epsilon_{w_k}) \leq -1$, otherwise $\chi_Q(\alpha, \epsilon_{w_k}) \geq 0$. In the local quiver there is a path from the vertex corresponding to W over the ones from S_{w_i} to S_v. Repeating the same argument starting from w_k we also have a path from the vertex from S_v over the ones from S_{w_j} to W.



The subquiver consisting of $\mathbf{1}$, ϵ_v and the two paths through the ϵ_{w_i} is reducible to . So if bot $\alpha_{w_1} > 1$ and $\alpha_{w_k} > 1$, (Q, α) is not coregular.

□

We will now look at the reduced quiver settings without loops.

Lemma 2.27. A quiver setting with dimension vector $\mathbf{1}$ is coregular if and only if the number of primitive cycles equals the dimension of $\mathbb{C}[\mathsf{iss}_\mathbf{1} Q]$.

Proof. The condition is obviously sufficient. It is also necessary because if the number of cycles is bigger than the dimension then there will be a relation between the cycles. If $\mathbb{C}[\mathsf{iss}_\mathbf{1} Q]$ is a polynomial ring, this relations must be of the form $Y = X_1 \dots X_k$ but this is impossible because Y is a primitive cycle. □

Lemma 2.28. A connected reduced quiver setting without loops is never coregular.

Proof. If $\alpha \neq \mathbf{1}$, consider the vertex v with the highest dimension. And look at the local quiver of the $(Q, \alpha - \epsilon_v) \oplus (Q, \epsilon_v)$. This is indeed a decomposition in simples and $\chi_Q(\alpha - \epsilon_v, \epsilon_v) = \chi(\alpha, \epsilon_v) - 1 < -1$.

Suppose thus $\alpha = \mathbf{1}$. Because (Q, α) is reduced, there are at least 2 arrows arriving and leaving every vertex. For a connected quiver without loops $\mathsf{Dim}\mathbb{C}[\mathsf{iss}_\mathbf{1} Q] = \#A - \#V + 1$ so we have to prove that for such quivers the number of primitive cycles is bigger than $\#A - \#V + 1$ or that Q constains a subquiver that is not coregular. We will do this by induction on the vertices.

- For $\#V = 2$ the statement is true because

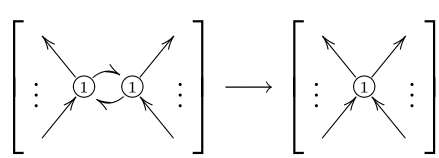
$$Q := \begin{tikzcd} \textcircled{1} \arrow[r, bend left, "k"] & \textcircled{1} \arrow[l, bend left, "l"] \end{tikzcd}, \ k, l \geq 2 \ \Rightarrow \ kl > k + l - 1.$$

- Suppose $\#V > 2$ and that we have a subquiver of the form

$$\begin{tikzcd} \textcircled{1} \arrow[r, bend left, "k"] & \textcircled{1} \arrow[l, bend left, "l"] \end{tikzcd} \ (*)$$

  If $k, l > 1$ we know that this subquiver and hence Q are not coregular.

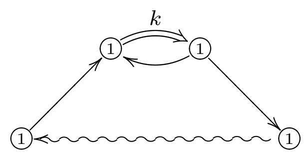
  If both k and l are 1 then substitute this subquiver by 1 vertex.



  The new quiver Q' is again reduced without loops because there are at least 4 arrows arriving in one of vertices of the subquiver and we only deleted 2, the same holds for the arrows leaving the subquiver. Q' has one primitive cycle less than the original. By induction we have that

$$\begin{aligned}\mathsf{Dim}\mathbb{C}[\mathsf{iss}_1 Q] &= \mathsf{Dim}\mathbb{C}[\mathsf{iss}_1 Q'] + 1\\ &> (\#A' - \#V' + 1) + 1\\ &= \#A - \#V + 1.\end{aligned}$$

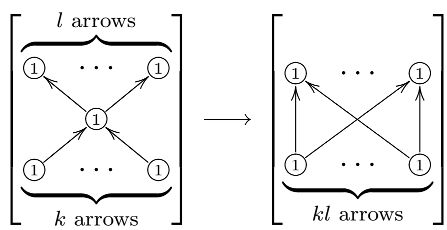
If for instance $k > 1$ then one can look at the subquiver of Q obtained by deleting the $k-1$ arrows, if this quiver is reduced then we are in the previous situation. If this is not the case Q contains a subquiver of the form



,

which is not coregular because it is reducible to $(*)$.

- If $\#V > 2$ and there are no subquivers of the form $(*)$, we can consider a random vertex v. Construct a new quiver Q' by making the following substitution for v



.

Q' is again reduced without loops and has the same number of primitive cycles, so by induction

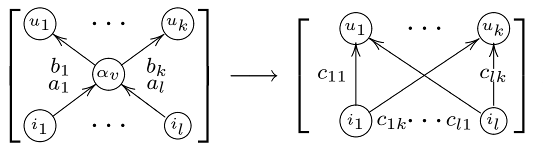
$$\begin{aligned}\mathsf{Dim}\mathbb{C}[\mathsf{iss}_1 Q] &= \mathsf{Dim}\mathbb{C}[\mathsf{iss}_1 Q']\\ &> \#A' - \#V' + 1\\ &= \#A + (kl - k - l) - \#V + 1 + 1\\ &> \#A - \#V + 1.\end{aligned}$$

$\square$

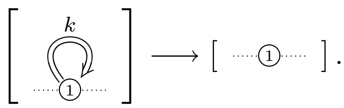
This concludes the proof of our classification result on coregular quiver settings.

Theorem 2.29. Let (Q, α) be a strongly connected quiver setting and (Q', α') is the quiver setting obtained after all possible reductions of the form

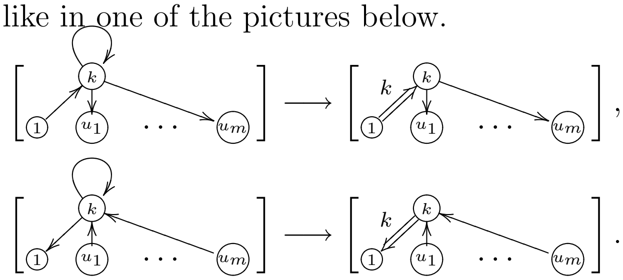
$\mathcal{R}_I$ If $\sum_{j=1}^{k} i_j \leq \alpha_v$ or $\sum_{j=1}^{l} u_j \leq \alpha_v$ we delete the vertex v.



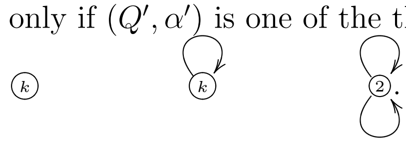
$\mathcal{R}_{II}$ Remove the loops on a vertex with dimension 1.



$\mathcal{R}_{III}$ Remove the only loop on a vertex with dimension $k > 1$ which has a neighborhood like in one of the pictures below.



(Q, α) is coregular if and only if (Q', α') is one of the three settings below:



Proof. This follows immediately from lemma's 2.26 and 2.28 and the fact that as proven in [22] the quiver settings that are listed in the theorem are coregular $\square$

2.4 Complete intersections

In this section we will use the techniques we introduced in the previous parts of this chapter to determine which quotient spaces of symmetric quiver settings are complete intersections.

Definition 2.30. A variety V of dimension n is called a complete intersection (C.I.) if

$$\mathbb{C}[V] \cong \mathbb{C}[X_1, \ldots, X_k]/(f_1, \ldots, f_l)$$

such that $k - l = n$.

A quiver setting (Q, α) is called a complete intersection (C.I.) if its corresponding quotient space $\mathsf{iss}_\alpha Q$ is.

Complete intersections are the simplest non-smooth varieties and from our point of view the property of $\mathsf{iss}_\alpha Q$ being a C.I. is much like being smooth.

It is trivial to prove that tensor products of complete intersections are again complete intersections. Therefore a connected sum of two quivers over a vertex with dimension one is a complete intersections if and only if the both quivers are C.I.'s.

If $\mathsf{iss}_\alpha Q$ is a complete intersection we have that the quotient space of a subquiver setting is again a complete intersection because we simply divide out some generators of $\mathbb{C}[\mathsf{iss}_\alpha Q]$. Also for every local quiver setting (Q_p, α_p) where $p \in \mathsf{iss}_\alpha Q$, the corresponding quotient space is a complete intersection if $\mathsf{iss}_\alpha Q$ is so (see [11]).

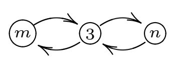
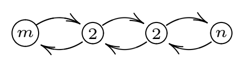
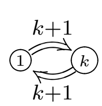
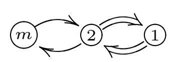
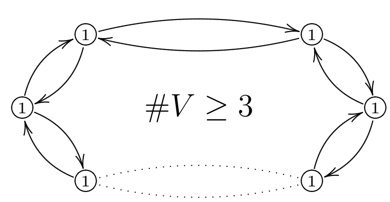
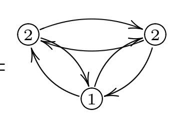
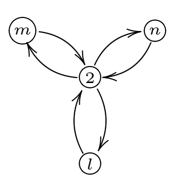
Another consequence is that the property of being a complete intersection is preserved under the three $\mathcal{R}$-moves and their inverses. If (Q, α) is a symmetric quiver setting, application of the moves $\mathcal{R}_I$ and $\mathcal{R}_{II}$ will give us again a symmetric setting. If we apply the $\mathcal{R}_{III}$-move this is not true anymore but in this case we always can apply an $\mathcal{R}_I$ to make it symmetric again. Therefore the notion of being reduced makes sense in the symmetric case.

To state our theorem more compact we introduce the notion *reduced without loops*, by this we mean that we apply the reduction steps $\mathcal{R}.$ as far as we can but at the end we remove all loops with $\mathcal{R}_I^{-1}$.

Theorem 2.31. Let (Q, α) be a symmetric quiver. $\mathsf{iss}_\alpha Q$ is a complete intersection if and only if Q can be written as a connected sum

$$Q = Q_1 \#_{v_1} \cdots \#_{v_r} Q_r \text{ with } \alpha_{v_r} = 1.$$

where the Q_i are quivers that are coregular or that can be reduced without loops to

- $I_a =$ , $I_b =$ ,
- $II_a =$  $II_b =$ 
- $III_a =$ , $III_b =$ 
- $IV =$  .

The structure of the theorem is much alike the symmetric coregular case. To prove this statement we only have to look at quiver settings that are not a connected sum of smaller components (these will be called *prime*) and that are reduced without loops. In what follows we will classify all such quivers that are complete intersections. This, together with the facts that being a C.I. is preserved under taking connected sums, taking subquivers and making reductions, will give us the proof of our theorem.

First of all we prove that the quiver setting $\textcircled{1} \mathrel{\substack{\Longrightarrow \\ \Longleftarrow}} \textcircled{1}$ is not a complete intersection (lemma 2.32). After that we look at two special quivers settings that are variants of cases II_b and III_b, (we denote these variants with a prime). We prove that these variants are a C.I. if and only if the are of the form II_b and III_b (lemmas 2.33-2.34.

In the second part of the proof we will study in depth all possible prime reduced quiver settings and prove that if they are a C.I. if they are of the forms $I-VI, II_b'$ or III_b' (lemmas 2.35-2.38). This is done by finding decompositions in simples that give rise to a local quiver that contains $\textcircled{1} \mathrel{\substack{\Longrightarrow \\ \Longleftarrow}} \textcircled{1}$. In the proofs we will only state the two components of the decomposition that are important and we will leave the calculation of $\chi_Q(\alpha_1, \alpha_2)$ to the reader.

Showing that I_a, I_b, III_a, IV are indeed a C.I. in combination with lemmas 2.33 and 2.34 finishes our proof.

Lemma 2.32. The quiver setting

$$\textcircled{m} \underset{k}{\overset{k}{\rightleftarrows}} \textcircled{n}, \;\; m \leq n, k > 1$$

has a quotient space which is a complete intersection if and only if $m = 1$ and $n \geq k - 1$.

Proof. If $m = 1$ the action on this vertex is trivial so we can see $\mathsf{Rep}_\alpha Q$ as the quiver situation

$$\textcircled{1} \xRightarrow{\;k\;} \textcircled{n} \xRightarrow{\;k\;} \textcircled{1}$$

$$\mathsf{GL}_n(\mathbb{C})$$

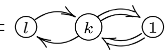
with only a $\mathsf{GL}_n(\mathbb{C})$ action on the central vertex. By 2.21 this is in fact the same as the space of $k \times k$-matrices with rank smaller than n. It is well known that this space is a complete intersection if and only if $k \geq n - 1$, see [7].

If both n, m are bigger than 1, we can make a decomposition

$$\textcircled{1} \rightleftarrows \textcircled{0} \oplus \textcircled{m-1} \rightleftarrows \textcircled{m} \oplus \cdots$$

with $\chi_Q(\alpha_1, \alpha_2) = 2m - m + 1 = m + 1 \geq 3$. $\square$

Lemma 2.33. The quiver setting

$$II'_b =$$  $$,\ k \geq 1, l = k, k-1$$

is a complete intersection if and only if $k = 2$.

Proof. For every representation of (Q, α) we define A to be the $2 \times k$-matrix corresponding to the double arrow to the left. B is the $k \times 2$ matrix coming from the double arrow to the right and C is the $k \times k$-matrix coming from the cycle starting in k running through l.

Using 2.1 we can check that $\mathbb{C}[\mathsf{iss}_\alpha Q]$ is generated by the following $5k$ invariants

$$X_i := \mathsf{Tr} C^i,\ 1 \leq i \leq k \text{ and } Y_j^{st}(AC^jB)_{st},\ 0 \leq j \leq k-1, s,t = 1,2.$$

Using 1.15 we can check that the dimension of $\mathsf{iss}_\alpha Q$ is

$$1 - \chi_Q(\alpha, \alpha) = \begin{cases} 4k & l = k \\ 4k & l = k-1 \end{cases}$$

So if $\mathsf{iss}_\alpha Q$ is a C.I. there must be exactly k relations if $k = l$ and $k+1$ if $l = k-1$.

- Case 1: $k = 2$. Suppose first that $l = 2$. We can deduce 2 simple relations

  $$\begin{aligned} \mathsf{Tr}(ABACB) = \mathsf{Tr}((AB)^2C) &= Tr(AB)\mathsf{Tr}(ACB) - \det(AB)\mathsf{Tr}(C) \\ \det(ACB) &= \det C \det(AB) \\ &\Rightarrow \\ \sum_{st} Y_0^{st} Y_1^{ts} &= \sum_{st} Y_0^{ss} Y_1^{tt} - X_1(Y_0^{11}Y_0^{22} - Y_0^{12}Y_0^{21})\ (i) \\ Y_1^{11}Y_1^{22} - Y_1^{12}Y_1^{21} &= \frac{1}{2}(X_1^2 - X_2)(Y_0^{11}Y_0^{22} - Y_0^{12}Y_0^{21})\ (ii) \end{aligned}$$

  Using the Groebner base algorithm in Maple we can check that these relations generate the ideal of relations. We know that $\mathsf{Dim}\ \mathsf{iss}_\alpha Q) = 8 = 10-2$ and hence (Q, α) is a complete intersection.

  For $k = 2$ and $l = 1$ the dimension of $\mathsf{Dim}\ \mathsf{iss}_\alpha Q$ is one lower and we have one extra relation $\det C = \frac{1}{2}(X_1^2 - X_2) = 0$. So if $k = 2$ $\mathsf{iss}_\alpha Q$ is always a complete intersection.

- Case 2:$k = 3$. If $k = 3$ we can produce relations similar to (i). Using the Caley-Hamilton identity in 3 dimensions for the sum of 2 matrices

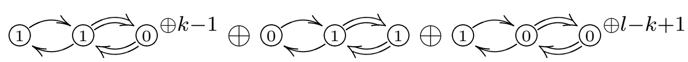
  $$\begin{aligned} &(M+N)^3 - \mathsf{Tr}(M+N)X^2 + \frac{1}{2}((\mathsf{Tr}(M+N))^2 - \mathsf{Tr}(M+N)^2)) \\ &\quad + \frac{1}{6}(\mathsf{Tr}(M+N)^3 - 3(\mathsf{Tr}(M+N))^3 + 2\mathsf{Tr}(M+N)\mathsf{Tr}(M+N)^2) = 0, \end{aligned}$$

we can express a MNM in terms of al other products of M and N with degree smaller or equal than 3. With this expression we can produce relations of the following forms

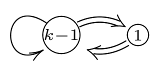
$$\begin{aligned}
\sum_{s,t} Y_1^{ts} Y_1^{st} &= Tr(A(CBAC)B) = \dots \\
&\quad \text{take } M = C \text{ and } N = BA \text{ and substitute } MNM \\
\sum_{s,t} Y_1^{ts} Y_2^{st} &= Tr(A(CBAC)CB) = \dots \\
&\quad \text{take } M = C \text{ and } N = BA \text{ and substitute } MNM \\
\sum_{s,t} Y_2^{ts} Y_2^{st} &= Tr(AC(CBAC)CB) = \dots \\
&\quad \text{take } M = C \text{ and } N = BA \text{ and substitute } MNM \\
\sum_{s,t} Y_2^{ts} Y_2^{st} &= Tr(A(C^2BAC^2)B) = \dots \\
&\quad \text{take } M = C^2 \text{ and } N = BA \text{ and substitute } MNM
\end{aligned}$$

Using Maple we can check that none of these relations is generated by the others. This together with the fact that there are no relations of smaller degree, shows us that for $k = l = 3$, $\mathsf{iss}_\alpha Q$ is not a complete intersection. The same can be deduced for $l = 2$.

- Case 3: $k > 3$. We can construct a decomposition of the form

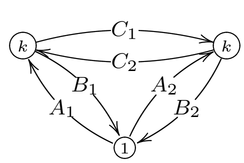
  The local quiver contains a subquiver of the form

  So if $\mathsf{iss}_\alpha Q$ is a C.I. for k then it is a C.I. for $k-1$. Continuing this construction we can reduce to the case $k = 3$. For $k = 3$, $\mathsf{iss}_\alpha Q$ is not a C.I. and therefore neither for $k > 3$.

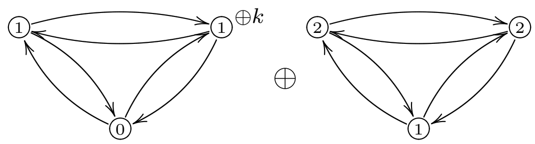
□

Lemma 2.34. The quiver setting

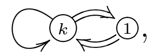
$III'_b =$ 

is a complete intersection if and only if $k \leq 2$.

Proof. If $k > 2$ then $\mathsf{iss}_\alpha Q$ cannot be a complete intersection because we can make the decompostion



of which the local quiver is

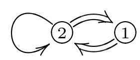


which is not a C.I. by lemma 2.33.

If $k = 2$ the ring of invariants is generated by the folowing traces

$$\begin{array}{rl} \mathsf{Tr}C_1C_2 & \mathsf{Tr}(C_1C_2)^2 \\ \mathsf{Tr}A_1B_1 & \mathsf{Tr}A_1C_1C_2B_1 \\ \mathsf{Tr}A_2B_2 & \mathsf{Tr}A_2C_1C_2B_2 \\ \mathsf{Tr}A_1C_1B_2 & \mathsf{Tr}A_1C_1C_2C_1B_2 \\ \mathsf{Tr}A_2C_2B_1 & \mathsf{Tr}A_2C_2C_1C_2B_1, \end{array}$$

where the matrices A_i, B_i and C_i correspond to the arrows drawn in the quiver setting. Consider the points in the open subset $U \subset \mathsf{iss}_\alpha Q$ for which $\det C_1C_2 \neq 0$. These points correspond to orbits that contain a representation for which C_1 is a unit matrix. Simplifying the traces, by setting $C_1 = 1$ and $C_2 = C$, we can consider $\mathbb{C}[\mathsf{iss}_\alpha Q]$ as a subring of the ring of invariants of the quiver setting



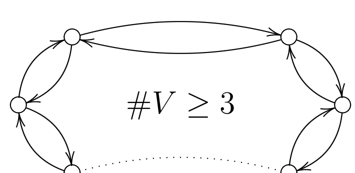
generated by

$$\begin{array}{r} X_1, X_2, Y_0^{11}, Y_1^{11} \\ Y_0^{12}, Y_1^{12}, Y_0^{21}, Y_1^{21} \\ Y_1^{22} \text{ and } X_1Y_1^{22} - \frac{1}{2}(X_1^2 - X_2)Y_0^{22} \end{array}$$

Using this fact for a Groebner base computation in Maple allows us again to prove that $\mathsf{iss}_\alpha Q$ is a C.I. $\square$

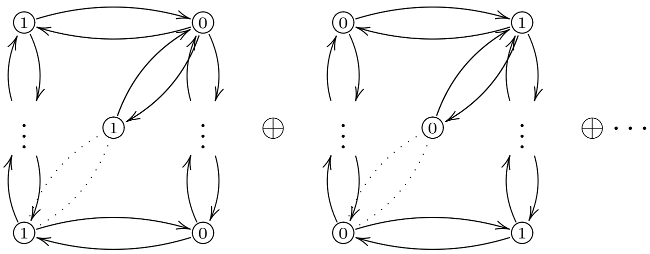
In the lemmas hereunder we assume that (Q, α) is a reduced prime symmetric quiver setting without loops for which $\mathsf{iss}_\alpha Q$ is a C.I.

Lemma 2.35. If Q contains a subquiver of the form



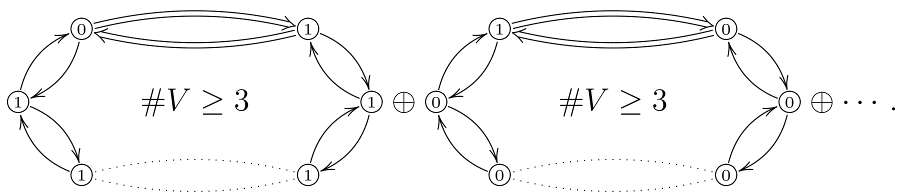
then (Q, α) is of the form III_a or III'_b.

Proof. If 2 vertices of the cycle in Q are connected to each other by a third way we can make a decomposition



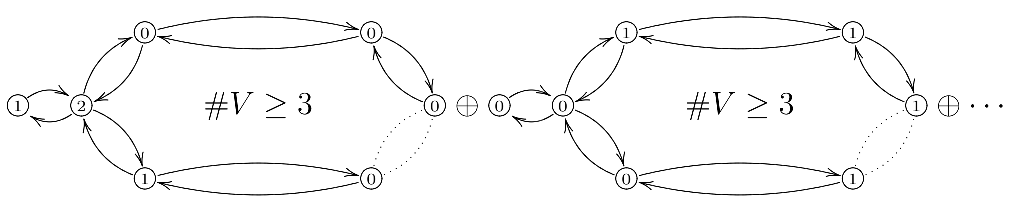
of which the locale quiver is not C.I., so this is not possible.

Also a situation where there are two vertices connected by a double arrow in both directions is impossible. At least one of those two vertices must have dimension one according to lemma 2.32, suppose the left one. Make a decomposition of the form



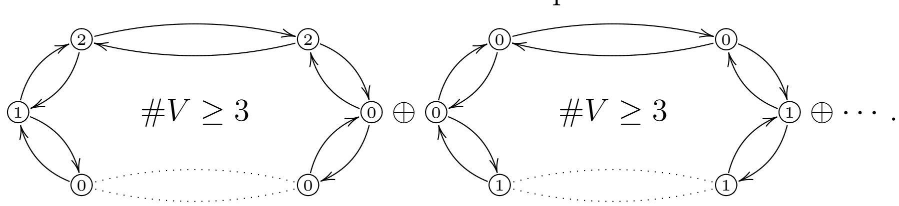
We remark that one of the vertices of the cycle must have dimension one otherwise we would have a decompositon containing two simples with dimension vector $\mathbf{1}$. Between those 2 components there are $\#A - \#V = \#V > 2$ arrows.

There cannot be any branching vertex with dimension $k > 1$ in the cycle. If there were, we could make a decomposition



This decomposition is always possible because if the second component contains only one vertex (i.e. in the case that the cycle has 3 vertices) the previous remark allows us to assume that this vertex has dimension 1 in our quiver setting.

If the cycle contains more than 3 vertices, the dimension vector must be $\mathbf{1}$. Otherwise, because of the reducedness, there must be two consecutive vertices with dimension 2 or more and we can make a decomposition of the form

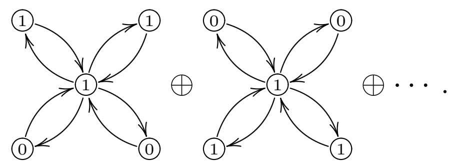


So if there are more than 3 vertices in the cycle we are in case III_a.

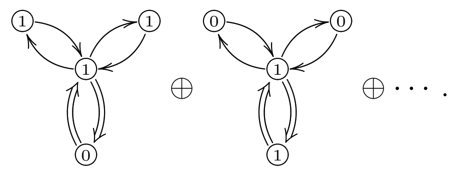
If there are 3 vertices and we are not in III_a, one of them must have dimension 1 the other must both have the same dimension because of the reducedness, so this is case III_b'. □

Lemma 2.36. If Q contains a branching vertex we are in case IV.

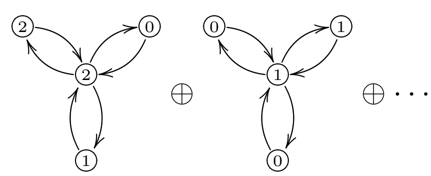
Proof. It is impossible that there are four vertices connected with the branching vertex. Because of 2.35 and the primeness of (Q, α), the branching vertex must have dimension 2 or more. Make a subdecomposition of the form



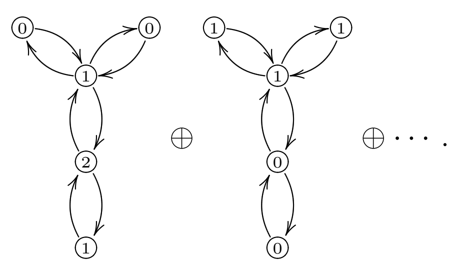
Also there is only one arrow to each vertex connected with the braching vertex because of the decomposition



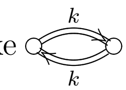
Suppose there are only 3 vertices connected with the branching vertex and its dimension is 3. Because (Q, α) is reduced one of the braching vertices must have dimension bigger than 1. We can make a decomposition of the form



If the branching vertex has dimension 2 then Q is of the form IV. If there would be another vertex connected to one of the outer vertices, this vertex would have dimension 2 or more because of the reducedness. Therefore we would have a decomposition like

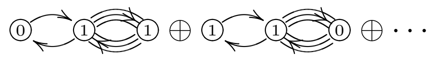


□

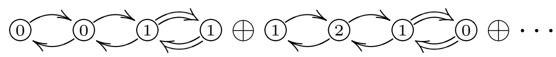
Lemma 2.37. If Q contains a quiver like  then we care in cases II_a or II_b

Proof. By lemma 2.32 we already know that one of the 2 vertices must have dimension one, and is because of the reducedness at the end of the quiver.

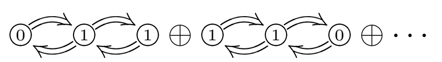
If $k > 2$ we can make a decomposition like



Suppose now that $k = 2$ then we are in situation II_a or II_b because if there are 4 vertices in Q we can make a decomposition like



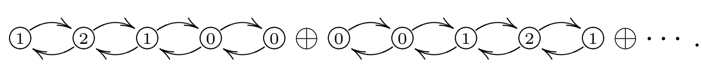
and if there are 3 vertices, only one double arrow can leave a vertex, otherwise we can make a decomposition like



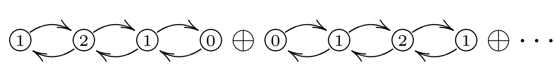
Together with 2.32 this implies that if Q has 2 vertices we are in case II_a and if Q has 3 vertices we are in II_b. $\square$

Lemma 2.38. If Q is linear, not containing double arrows then we are in cases I_a, I_b or coregular.

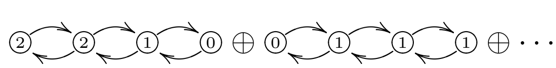
Proof. If Q is linear and prime, Q cannot have five vertices. Otherwise we would have a subdecomposition like



Take Q with 4 vertices and consider the vertex with the highest dimension to be the second from the left (this is always possible because reducedness prohibits it to sit at the end). If this dimension is bigger than 2 than we can make a decomposition like

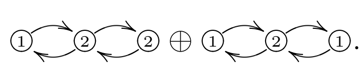


or



Depending on whether the first vertex has dimension 1 or higher. So if Q has 4 vertices we are in case I_b

If Q has 3 vertices the central vertex has the highest dimension. In order to be reduced, none of the vertices can have dimension 1 (otherwise we could apply move $\mathcal{R}_{III}$). If this dimension is bigger than 3 we can make a decomposition like



So in this case, if the dimension of the central vertex is 3 we are in I_a, if it is 2 then (Q, α) is coregular.

If Q has 2 vertices Q is coregular. $\square$

Finally we only have to prove that the listed quiver settings have indeed a quotient space that is a C.I.

For II_a this is already done in lemma 2.32, for II_b and III_b in lemmas 2.33 and 2.34.

In [12] it is proven that the ring of invariants of the quiver situation IV where $l, n, m \geq 2$, can be seen as a free module of rank two over the subalgebra generated by the invariants

$$TrX_i, TrX_iX_j,\ 1 \leq i \leq j \leq 3$$

In this notation X_i stand for the matrix coming from the path that runs through one of the three branches.

This subalgebra is a polynomial ring and the element of rank two is $\mathsf{Tr}X_1X_2X_3$ and satisfies the equation

$$(\mathsf{Tr}X_1X_2X_3)^2 + A(\mathsf{Tr}X_1X_2X_3) + B = 0 \quad (\dagger)$$

where

$$\begin{aligned}
A &= \mathsf{Tr}X_1\mathsf{Tr}X_2X_3 + \mathsf{Tr}X_2\mathsf{Tr}X_3X_1 + \mathsf{Tr}X_3\mathsf{Tr}X_1X_2 - \mathsf{Tr}X_1\mathsf{Tr}X_2\mathsf{Tr}X_3 \\
B &= \det X_1(\mathsf{Tr}X_2X_3)^2 + \det X_2(\mathsf{Tr}X_3X_1)^2 + \det X_3(\mathsf{Tr}X_1X_2)^2 \\
&\quad - \mathsf{Tr}X_1\mathsf{Tr}X_2\mathsf{Tr}X_1X_2 \det X_3 - \mathsf{Tr}X_2\mathsf{Tr}X_3\mathsf{Tr}X_2X_3 \det X_1 \\
&\quad - \mathsf{Tr}X_3\mathsf{Tr}X_1\mathsf{Tr}X_3X_1 \det X_2 \\
&\quad + (\mathsf{Tr}X_1)^2 \det X_2 \det X_3 + (\mathsf{Tr}X_2)^2 \det X_3 \det X_1 + (\mathsf{Tr}X_3)^2 \det X_1 \det X_2 \\
&\quad - 4 \det X_1 \det X_2 \det X_3 + \mathsf{Tr}X_1X_2\mathsf{Tr}X_2X_3\mathsf{Tr}X_3X_1
\end{aligned}$$

and $\det X_i$ stands for $\frac{1}{2}((\mathsf{Tr}X_i)^2 - \mathsf{Tr}X_i^2)$. So for $l, m, n \geq 2$, situation VI is definitely a complete intersection. For the dimension vectors where l, m or n equal 1, the only extra relations we have to divide out are of the form $\mathsf{Tr}X_i^2 = (\mathsf{Tr}X_i)^2$, so for situation IV, $\mathsf{iss}_\alpha Q$ is always a complete intersection.

Using this explicit expression one can also easily deduce that the subring

$$\mathbb{C}[\mathsf{Tr}X_i, \mathsf{Tr}X_iX_j,\ 1 \leq i \leq j \leq 3, (i,j) \neq (a,b)][\mathsf{Tr}X_1X_2X_3]$$

is a polynomial ring for every couple (a, b) (A relation in this ring combined with $(\dagger)$ would imply that either $\mathsf{Tr}X_1X_2X_3$ satisfies a linear equation or

$$\mathbb{C}[\mathsf{Tr}X_i, \mathsf{Tr}X_iX_j,\ 1 \leq i \leq j \leq 3]$$

is not polynomial. This fact can be used to prove that I_b is also a complete intersection.

Suppose first that m, n are bigger than 1, then we can modify setting I_b to the following situation

$$\begin{tikzcd} ② \arrow[loop left, "A"] \arrow[r, bend left, "C"] & ② \arrow[loop right, "B"] \arrow[l, bend left, "D"] \end{tikzcd} .$$

If $A, \ldots, D$ are the matrices that represent the corresponding arrows then we can make a list of generators:

$$\begin{array}{rl} \mathsf{Tr}A & \mathsf{Tr}A^2 \\ \mathsf{Tr}B & \mathsf{Tr}B^2 \\ \mathsf{Tr}CD & \mathsf{Tr}(CD)^2 \\ \mathsf{Tr}CAD & \mathsf{Tr}DBC \\ \mathsf{Tr}CADB & \mathsf{Tr}CADCDB. \end{array}$$

Every other trace of cycles can be written in function of those ten using the Caley-Hamilton identity for A, B or CD.

The first nine traces generate a polynomial algebra. If there would be an algebraic relation between those traces, we could specialize this to $C = 1$ to obtain an algebraic relation in

$$\mathbb{C}[TrX_i, TrX_iX_j,\ 1 \leq i \leq j \leq 3, (i,j) \neq (1,2)][\mathsf{Tr}X_1X_2X_3]$$

where we set $X_1 = A$, $X_2 = B$, $X_3 = D$, which is impossible.

To find a quadratic relation for $\mathsf{Tr}CADCDB$ we use (†) and specialize this to $X_1 = CAD$, $X_2 = CD$ and $X_3 = B$. So again the ring of invariants is a rank 2 free module over a polynomial ring and hence I_b is a complete intersection when $m, n \geq 2$. For $m = 1$ or $n = 1$ we only have to divide out $\mathsf{Tr}A^2 = (\mathsf{Tr}A)^2$ or $\mathsf{Tr}B^2 = (\mathsf{Tr}B)^2$ so in theses cases we will also have a C.I.

To prove that I_a is a complete intersection, we can use the same technique as for IV. In [25] it was proven that the ring of invariants of I_a can also be seen as a rank 2 free module over the subalgebra generated by

$$TrX_1^j, TrX_2^j, TrX_1^sX_2^t,\ 1 \leq j \leq 3, 1 \leq s,t \leq 2.$$

where X_1 and X_2 are similar to the previous situation. This ring is again polynomial for all possible m, n.

Finally case III_a is a C.I. because its dimension is $\#V + 1$ and its quotient ring is

$$\mathbb{C}[X_1, \ldots, X_k, Y_+, Y_-]/(X_1 \cdots X_k - Y_+Y_-)$$

Where the X_i stand for traces of the small cycles between 2 vertices and Y_+, Y_- are the big cycles clock- and anti clockwise.

Chapter 3

Moduli Spaces of Quivers

In the first chapter we have seen that we can classify the semisimple α-dimensional isomorhpism classes of representations of a quiver Q in a variety $\mathsf{iss}_\alpha Q$. The ring of polynomial functions over this variety is generated by traces of cycles in Q. If Q has no cycles at all this means that this ring is just the field of complex numbers and hence $\mathsf{iss}_\alpha Q$ contains only one semisimple representation namely the zero representation.

If one wants to classify all representations in this case, one has to look at a bigger ring than the ring of invariants. In this chapter we will look at the ring of semi-invariants which will give us new varieties of quiver representations, the moduli spaces of θ-semistable representation. Moduli spaces for finite dimensional algebras and quivers in genreral are studied in the article [15] by Alastair King. The technique of determinantal invariants for quivers in the next section, has been obtained independantly by various people among which Schofield and Vandenbergh [23], Domokos and Zubkov [10], and Derksen and Weyman [9]. We will primarily follow the outline as in sections 6.3, 7.2 and 7.3 of [20].

3.1 Moduli spaces of Quiver representations

Instead of looking at invariant functions one can broaden our view to functions that are not invariant under the GL_α-action, but almost invariant. In this section we suppose that (Q, α) is a quiver setting with k vertices.

Definition 3.1. A character of GL_α is a group homomorphism $\chi : \mathsf{GL}_\alpha \to \mathbb{C}^*$. For every k-tuple $\theta := (t_1, \ldots, t_k) \in \mathbb{Z}^k$ we can construct a character

$$\chi_\theta : \mathsf{GL}_\alpha \to \mathbb{C}^* : (g_1, \ldots, g_k) \mapsto \mathsf{Det}(g_1)^{t_1} \cdots \mathsf{Det}(g_k)^{t_k}.$$

One can easily prove that all characters of GL_α are of this form.

A polynomial function $f \in \mathsf{Rep}_\alpha Q$ is called a *semi-invariant* of weight θ if and only if

$$\forall W \in \mathsf{Rep}_\alpha Q : \forall g \in \mathsf{GL}_\alpha : f(gW) = \chi(g)f(W)$$

Semi-invariants of a certain weight do not form a ring. To remedy this problem we will use a trick. Consider the vector space $\mathsf{Rep}_\alpha Q \oplus \mathbb{C}$ with the following action of GL_α

$$g(W, c) := (gW, \chi_\theta^{-1}(g)c).$$

The ring of polynomial functions over this space is $\mathbb{C}[\mathsf{Rep}_\alpha Q][X]$ and can be given a gradation by defining $\deg f = 0$ if $f \in \mathbb{C}[\mathsf{Rep}_\alpha Q]$ and $\deg X = 1$. The action preserves this gradation so the corresponding ring of invariants, $\mathbb{C}[\mathsf{Rep}_\alpha Q][X]^{\mathsf{GL}_\alpha}$ is also a graded ring. This ring consists of functions of the form

$$f := f_0 + f_1 X + \cdots + f_m X^m,$$

where f_i is a semi-invariant of weight $\chi_{i\theta}$. Because $\mathbb{C}[\mathsf{Rep}_\alpha Q][X]^{\mathsf{GL}_\alpha}$ is graded we can look at the corresponding projective variety.

Definition 3.2. For a quiver Q with n vertices, a dimension vector α and a character $\theta \in \mathbb{Z}^n$ we define the *moduli space of semistable quiver representations of dimension* α as

$$\mathsf{M}_\alpha^{\mathsf{ss}} Q := \mathsf{Proj}\ \mathbb{C}[\mathsf{Rep}_\alpha Q][X]^{\mathsf{GL}_\alpha}$$

We say that a representation W is χ_θ-semistable if and only if there exists a semi-invariant f of weight $\chi_{i\theta}, i \in \mathbb{N}$ such that $f(W) \neq 0$. We will denote the subset of all such representations as $\mathsf{Rep}_\alpha^{\mathsf{ss}}(Q, \theta)$. $\mathsf{Rep}_\alpha^{\mathsf{ss}}(Q, \theta)$ is a Zariski-open subset of $\mathsf{Rep}_\alpha Q$ but it can be empty for certain θ's.

One can prove the folowing theorem see [15]:

Theorem 3.3. The moduli space $\mathsf{M}_\alpha^{\mathsf{ss}}(Q, \theta)$ classifies all closed GL_α-orbits in $\mathsf{Rep}_\alpha^{\mathsf{ss}}(Q, \theta)$.

The concept of semistability is defined from an invariant theoretic point of view, which is not easy to work with. Fortunately it can be translated to a representation theoretic property in the following way

Definition 3.4. For a quiver setting (Q, α) with n vertices and a vector $\theta \in \mathbb{Z}^n$ we call a representation $W \in \mathsf{Rep}_\alpha Q$

1. θ-semistable if and only if $\alpha \cdot \theta = 0$ and for every subrepresentation W' with dimension vector β we have that $\theta \cdot \beta \geq 0$.

2. θ-stable if and only if $\alpha \cdot \theta = 0$ and for every non-trivial subrepresentation W' (i.e. $W' \neq 0$ and $W' \neq W$) with dimension vector β we have that $\theta \cdot \beta > 0$.

Theorem 3.5.

1. A representation is χ_θ-semistable if and only if it is θ-semistable.

2. The closed GL_α-orbits in $\mathsf{Rep}^{\mathsf{ss}}_\alpha(Q,\theta)$ are exactly the representation classes that decompose as a direct sum of θ-stable representation classes.

Proof. see [15] □

To study $\mathsf{M}^{\mathsf{ss}}_\alpha(Q,\theta)$ we first need a description of the graded ring associated to this moduli space. Therefore we search for a set of generators for this ring.

There exists an easy recipe for semi-invariants of weight θ. Take two dimension vectors ξ, η such that $\theta = \xi - \eta$ and construct two matrices over the algebra $\mathbb{C}Q$.

$$\Xi := \begin{pmatrix} \underbrace{\begin{matrix} v_1 & & \\ & \ddots & \\ & & v_1 \end{matrix}}_{\xi_{v_1}\,times} & & \\ & \ddots & \\ & & \underbrace{\begin{matrix} v_k & & \\ & \ddots & \\ & & v_k \end{matrix}}_{\xi_{v_k}\,times} \end{pmatrix}, \qquad H := \begin{pmatrix} \underbrace{\begin{matrix} v_1 & & \\ & \ddots & \\ & & v_1 \end{matrix}}_{\eta_{v_1}\,times} & & \\ & \ddots & \\ & & \underbrace{\begin{matrix} v_k & & \\ & \ddots & \\ & & v_k \end{matrix}}_{\eta_{v_k}\,times} \end{pmatrix}$$

These two diagonal matrices of sizes $|\xi|$ and $|\eta|$ can be used to construct a vecor space of matrices:

$$\mathcal{S}_{\xi,\eta} := \Xi \cdot (\mathsf{Mat}_{|\xi|\times|\eta|}(\mathbb{C}Q)) \cdot H$$

Every element of $\mathcal{M} \in \mathcal{S}_{\xi,\eta}$ consists of a matrix of which the entries are elements from spaces of the form $v_i \mathbb{C}Q v_j$. If we have a representation $W \in \mathsf{Rep}_\alpha Q$, every matrix element of $\mathcal{M}$ represents an $\alpha_{v_i} \times \alpha_{v_j}$-dimensional complex matrix. Putting all these little matrices together we can turn $\mathcal{M}$ into a big matrix $\mathcal{M}_W$ with dimensions

$$\sum_{v\in V} \xi_v \alpha_v \;\times\; \sum_{v\in V} \eta_v \alpha_v.$$

If W is semistable we have that $\theta \cdot \alpha = 0$, so using $\theta = \xi - \eta$ this matrix must be square, so we can take the determinant of it. The map

$$f_{\mathcal{M}} : \mathsf{Rep}^{\mathsf{ss}}_\alpha(Q,\theta) \to \mathbb{C} : W \mapsto \mathsf{Det}\mathcal{M}_W$$

will give us a semi-invariant of character θ because

$$
\begin{aligned}
f_{\mathcal{M}}(gW) &:= \mathsf{Det}\mathcal{M}_{gW} \\
&= \mathsf{Det}\begin{pmatrix} \underbrace{\begin{matrix} g_{v_1} & & \\ & \ddots & \\ & & g_{v_1}\end{matrix}}_{\xi_{v_1} times} & & \\ & \ddots & \\ & & \underbrace{\begin{matrix} g_{v_k} & & \\ & \ddots & \\ & & g_{v_k}\end{matrix}}_{\xi_{v_k} times} \end{pmatrix} \mathcal{M}_W \begin{pmatrix} \underbrace{\begin{matrix} g_{v_1}^{-1} & & \\ & \ddots & \\ & & g_{v_1}^{-1}\end{matrix}}_{\eta_{v_1} times} & & \\ & \ddots & \\ & & \underbrace{\begin{matrix} g_{v_k}^{-1} & & \\ & \ddots & \\ & & g_{v_k}^{-1}\end{matrix}}_{\eta_{v_k} times} \end{pmatrix} \\
&= \prod_{v\in V} \mathsf{Det} g_v^{\xi_v} \mathsf{Det}\mathcal{M}_W \prod_{v\in V} \mathsf{Det} g_v^{-\eta_v} \\
&= \chi_\theta(g) f_{\mathcal{M}}(W).
\end{aligned}
$$

Such a semi-invariant is called a *determinantal* semi-invariant. These invariants suffice to describe our ring of semi-invariants.

Theorem 3.6. $\mathbb{C}[\mathsf{Rep}_\alpha Q][X]^{\mathsf{GL}_\alpha}$ is generated by traces of cycles and elements of the form $f_{\mathcal{M}}X^i$ where $f_{\mathcal{M}}$ is a determinantal semi-invariant of weight $i\theta$.

Proof. see [23] □

Every projective variety can locally be seen as an affine variety in the following way. One takes a homogeneous element and one considers the projective localization of the graded ring by this element. This new ring corresponds to an affine variety which can be seen as a Zarisky-open subset of our projective variety. In our case we can take a determinantal semi-invariant $f_{\mathcal{M}}$ of weight $i\theta$ and localize $\mathbb{C}[\mathsf{Rep}_\alpha Q][X]^{\mathsf{GL}_\alpha}$ by the element $f_{\mathcal{M}}X^i$. It would be very convenient if we could see the corresponding affine variety

$$\mathsf{Specm}\mathbb{C}[\mathsf{Rep}_\alpha Q][X]^{\mathsf{GL}_\alpha}_{f_{\mathcal{M}}X^i}$$

as the representation variety of a certain new algebra. This is indeed possible. For every matrix $\mathcal{M} \in \mathcal{S}_{\xi,\eta}$ we can construct the universal localization of $\mathbb{C}Q$ by $\mathcal{M}$

Definition 3.7. The universal localization of $\mathbb{C}Q$ by $\mathcal{M}$, denoted by $\mathbb{C}Q_{\mathcal{M}}$, is the quotient of the free product

$$\mathbb{C}Q * \mathbb{C}\langle y_{ij}, 1 \le i \le |\eta|, 1 \le j \le |\xi|\rangle.$$

by the ideal generated by the relations determined by the matrix equations

$$\mathcal{M}\cdot Y = \Xi,\ Y\cdot\mathcal{M} = H,$$

where Ξ and H are defined as above and Y is the matrix (y_{ij}).

It is easy to verify that the representation variety $\mathsf{Rep}_n\mathbb{C}Q_{\mathcal{M}}$ can be seen as the open affine subset of $\mathsf{Rep}_n\mathbb{C}Q$ of representations W for which the matrix $\mathcal{M}_W$ is invertible. Therefore one can speak about the dimension vector of a representation $W \in \mathsf{Rep}_n\mathbb{C}Q_{\mathcal{M}}$ and hence one can consider the representation spaces $\mathsf{Rep}_\alpha\mathbb{C}Q_{\mathcal{M}}$ and $\mathsf{iss}_\alpha\mathbb{C}Q_{\mathcal{M}} := \mathsf{Rep}_\alpha\mathbb{C}Q_{\mathcal{M}}/\mathsf{GL}_\alpha$.

Theorem 3.8.

1. Suppose $W \in \mathsf{Rep}_\alpha Q$ is a representation for which $f_{\mathcal{M}}(W) \neq 0$. W is θ-stable if and only if W corresponds to a simple representation of $\mathbb{C}Q_{\mathcal{M}}$.
2. $\mathsf{Specm}\mathbb{C}[\mathsf{Rep}_\alpha Q][X]^{\mathsf{GL}_\alpha}_{f_{\mathcal{M}}X^i} \cong \mathsf{iss}_\alpha\mathbb{C}Q_{\mathcal{M}}$.
3. $\mathsf{M}^{\mathsf{ss}}_\alpha(Q,\theta)$ can be covered by affine spaces of the form $\mathsf{iss}_\alpha\mathbb{C}Q_{\mathcal{M}}$.

Proof. See [20, Theorem 7.9] □

So if we want to look at the local structure of $\mathsf{M}^{\mathsf{ss}}_\alpha(Q,\theta)$ around a point p that corresponds to a direct sum of θ-stable representations

$$W := S_1^{\oplus a_1} \oplus \cdots \oplus S_k^{\oplus a_k},\ (S_i \text{ has dimension vector} \alpha_i).$$

Because W is semi-stable we have a semi-invariant $f_{\mathcal{M}}$ for which $f_{\mathcal{M}}(W)$ is not zero. We can look at the corresponding representation of $\mathbb{C}Q_{\mathcal{M}}$. This has of course a similar decomposition but now in simples. Therefore we can construct the local quiver setting (Q_p, α_p) of this representation class. The number of arrows in between two vertices of Q_p is $\mathsf{DimExt}_{\mathbb{C}Q_{\mathcal{M}}}(S_i, S_j)$ where S_i and S_j are seen as the simple representations of $\mathbb{C}Q_{\mathcal{M}}$. However one can prove that the category of $\mathbb{C}Q_{\mathcal{M}}$-modules is closed under extensions in the category of $\mathbb{C}Q$-modules. Therefore

$$\mathsf{Ext}_{\mathbb{C}Q_{\mathcal{M}}}(S_i, S_j) \cong \mathsf{Ext}_{\mathbb{C}Q}(S_i, S_j).$$

The dimension of the last extension space can be calculated with the Euler form of Q. So we can conclude this section by formulating the local quiver theorem for moduli spaces as is stated in [2]

Theorem 3.9. For every point $p \in \mathsf{M}^{\mathsf{ss}}_\alpha(Q,\theta)$, corresponding to a θ-semistable quiver representation

$$S_1^{\oplus a_1} \oplus \cdots \oplus S_k^{\oplus a_k},\ (S_i \text{ has dimension vector} \alpha_i)$$

we have an étale isomophism between an open neighborhood of the zero representation in $\mathsf{iss}_{\alpha_p}Q_p$ and an open neighborhood of p. Q_p has k vertices and between the i^{th} and the j^{th} vertex there are

$$\delta_{ij} - \chi_Q(\alpha_i, \alpha_j)$$

The dimension vector α_p assigns to the i^{th} vertex dimension a_i.

In the next sections we will illustrate this theorem for some examples, and use the results of the previous chapter to find some smooth moduli spaces. The outline follows the article [1].

3.2 Free products and Bipartite quivers

3.2.1 Introduction

Consider a cylinder with q line segments on its surface, equidistant and parallel to its axis. If the ends of this cylinder are identified with a twist $2\pi\frac{p}{q}$ where p is an integer relatively prime to q, one obtains a single curve on the surface of a torus. Such a curve is called a torus knot, and is denoted by $K_{p,q}$. The fundamental group of the complement $\mathbb{R}^3\backslash K_{p,q}$ is called the (p,q)-torus knot group and is equivalent to the group

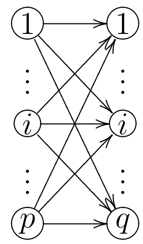
$$\langle x, y \mid x^p = y^q \rangle.$$

The center of this torus knot group is generated by the element y^q, so the quotient of a torus knot group with its center is equivalent to the free product $\mathbb{Z}_p * \mathbb{Z}_q$. If one wants to study irreducible representations of such a torus knot group, it suffices to study the representation theory of the quotient, $\mathbb{Z}_p * \mathbb{Z}_q$. In [2], Adriaenssens and Le Bruyn show that one can reduce the complex representation theory of the free product of two finite cyclic groups to the representation theory of a certain bipartite quiver.

The equivalence between representations of $\mathbb{Z}_p * \mathbb{Z}_q$ and representations of quivers is achieved as follows (see also [**?**]). Consider a complex representation V of the free product. By looking only at the action of $\mathbb{Z}_p$, one can decompose the vector space V into a direct sum of eigenspaces $V_{\xi^1} \oplus \cdots \oplus V_{\xi^p}$ where ξ is a pth root of unity. Repeating this for $\mathbb{Z}_q$, we obtain a double decomposition:

$$\begin{tikzcd}
V_{\xi^1} \oplus \cdots \oplus V_{\xi^p} \arrow[r, "\cong"] & V \arrow[r, "\cong"] & W_{\eta^1} \oplus \cdots \oplus W_{\eta^q} \arrow[d] \\
V_{\xi^i} \arrow[u, hook] \arrow[rr, "m_{ij}"'] & & W_{\eta^j}
\end{tikzcd}$$

So the canonical situation is that we have $p+q$ vectorspaces and a linear map from each of the p first spaces to each of the q last. This is in fact a representation of the following quiver:



The only restriction on the maps is that they must add up to an invertible map M between $V_{\xi^1} \oplus \cdots \oplus V_{\xi^p}$ and $W_{\eta^1} \oplus \cdots \oplus W_{\eta^q}$, because all the maps actually are restrictions of the indentity on V. This condition is necessary and sufficient. If we define the dimension vector of a $\mathbb{Z}_p * \mathbb{Z}_q$-representation as the vector

$$\alpha := (\mathsf{Dim} V_{\xi^1}, \ldots, \mathsf{Dim} V_{\xi^p}; \mathsf{Dim} W_{\eta^1}, \ldots, \mathsf{Dim} W_{\eta^q}),$$

we can say that there is an equivalence of categories between the category

$$\mathsf{Rep}_\alpha \mathbb{Z}_p * \mathbb{Z}_q$$

containing the representations with dimension vector α and the Zariski open subset U of $\mathsf{Rep}_\alpha Q$, consisting of the α-dimensional representations of the quiver for which the block matrix M is invertible. The action of $\mathsf{GL_n}$ on $\mathsf{Rep}_\alpha \mathbb{Z}_p * \mathbb{Z}_q$ translates itself into an action of $\mathsf{GL}_\alpha = \prod_i \mathsf{GL}_{\alpha_i}$ on $\mathsf{Rep}_\alpha Q$. So classifying the representation classes in $\mathsf{Rep}_\alpha \mathbb{Z}_p * \mathbb{Z}_q$ is the same as classifying the orbit of the GL_α-action in U. Doing this we will find that $\mathsf{iss}_\alpha \mathbb{Z}_p * \mathbb{Z}_q \equiv U/\mathsf{GL}_\alpha$ is an affine variety containing the semisimple representation classes of $\mathbb{Z}_p * \mathbb{Z}_q$.

A geometrically more appealing approach to study this affine variety is to look at a certain projective closure of this variety: the moduli space of α-dimensional θ-semistable representations of the quiver. If we take θ to be the vector

$$(-1, \ldots, -1; 1, \ldots, 1) \in \mathbb{Z}^{p+q},$$

it is easy to verify that every representation in U is in fact θ-semistable. Indeed, every representation $V \in \mathbb{Z}_p * \mathbb{Z}_q$ is in fact a representation in $\mathsf{Rep}_\alpha Q$ for which the matrix M is invertible. This matrix can be seen as $\mathcal{M}_V$ where

$$\mathcal{M} := (\textcircled{\scriptsize i} \longrightarrow \textcircled{\scriptsize j})_{ij} \in \mathsf{Mat}_{p \times q} \mathbb{C}Q.$$

This means that $f_{\mathcal{M}}(V) \neq 0$ and we can conclude that V is a representation of the universal localization $\mathbb{C}Q_{\mathcal{M}}$ and consequently we have the following diagram:

$$\begin{tikzcd}
\mathsf{Rep}_\alpha Z_p * \mathbb{Z}_q = \mathsf{Rep}_\alpha \mathbb{C}Q_{\mathcal{M}} \arrow[r, hook] \arrow[d, two heads] & \mathsf{Rep}^{ss}_\alpha(Q, \theta) \arrow[d, two heads] \\
\mathsf{iss}_\alpha Z_p * \mathbb{Z}_q = \mathsf{iss}_\alpha \mathbb{C}Q_{\mathcal{M}} \arrow[r, hook] & \mathsf{M}^{\mathsf{ss}}_\alpha(Q, \theta)
\end{tikzcd}$$

This diagram indicates that to study the representations of $\mathbb{Z}_p * \mathbb{Z}_q$, one could first try to study the moduli space $M^{ss}_\alpha(Q,\theta)$.

From now on we are going to work exclusively with θ-semistable representations of the quiver Q, so a notation has to be fixed. The vector spaces on each vertex will be denoted by $V_i, i = 1,\dots,p$ for the left vertices of the quiver, and $W_i, i = 1,\dots,q$ for the right ones.

The semistability implies that the dimension vector is of the following form

$$\alpha := (a_1,\dots,a_p;b_1,\dots,b_q) \text{ where } \sum_{i=1}^p a_i = \sum_{i=1}^q b_i =: n.$$

If we look at the Euler form of the quiver Q, i.e. the matrix with entries

$$[\chi_Q]_{ij} := \delta_{ij} - \#\{\text{arrows from vertex } i \text{ to } j\ \},$$

we can decompose it to a block matrix of the following form

$$\left[\begin{array}{ccc|ccc}
1 & & 0 & -1 & \dots & -1 \\
 & \ddots & & \vdots & & \vdots \\
0 & & 1 & -1 & \dots & -1 \\
\hline
0 & \dots & 0 & 1 & & 0 \\
\vdots & & \vdots & & \ddots & \\
0 & \dots & 0 & 0 & & 1
\end{array}\right]$$

Now consider two dimension vectors α_1 and α_2. One can easily compute their image under the Euler form:

$$\chi_Q(\alpha_1,\alpha_2) = \alpha_1 \cdot \alpha_2 - n_1 n_2,$$

where n_1 resp. n_2 equals the sum of the first p entries of α_1 resp. the last q entries of α_2. For the remainder of this section, n shall always equal the sum of the first p (or the last q) entries of a semistable dimension vector considered.

The last convention we make is that we will write elements of GL_α as follows:

$$g := (g_1,\dots,g_p;h_1,\dots,h_q)\ g_i \in \mathsf{GL}_{a_i}, h_i \in \mathsf{GL}_{b_i}.$$

We now have almost everything we need to determine which moduli spaces are smooth projective varieties. The only things left to know are the θ-(semi)stable representations of our quiver. These were determined by Adriaenssens and Le Bruyn in [2].

Theorem 3.10.

1. For a dimension vector $\alpha = (a_1, \dots, a_p; b_1, \dots, b_q)$ such that $\theta \cdot \alpha = 0$, there always exist θ-semistable representations, in this case we denote $n = \sum_{i=1}^{p} a_i$.
2. $\mathsf{M}^{\mathsf{ss}}_\alpha(Q, \theta)$ contains a non-empty subset of θ-stable representations, which is then a dense open subset, if and only if

$$\forall i \le p, j \le q : a_i + b_j \le n \ (*)$$

unless $\forall i, j : a_i = b_j$ or $n = 1$ in which case $\mathsf{M}^{\mathsf{ss}}_\alpha(Q, \theta)$ is just a point.

Because we use the condition $(*)$ quite often in the next section we will call a dimension vector satisfying $(*)$ *almost simple*.

3.2.2 Smoothness of $\mathsf{M}^{\mathsf{ss}}_\alpha(Q, \theta)$

In this section we use the local quivers introduced earlier to determine which of the moduli spaces correspond to a smooth projective variety.

If $\mathsf{M}^{\mathsf{ss}}_\alpha(Q, \theta)$ is a smooth space, it will be definitely smooth in the semisimple points which have only two factors with multiplicity 1. We will see that this is not the case for most of the moduli spaces. By the lemma ... we only have to check that the number of arrows connecting both factors is not greater than 1, i.e.

$$\chi_Q(\alpha_1, \alpha_2) \ge -1.$$

This fact enables us to deduce the following

Lemma 3.11. Suppose $\alpha = (a_1, \dots, a_p; b_1, \dots, b_q)$ is a simple dimension vector, then the degeneration

$$(a_1, \dots, a_i - 1, \dots, a_p; b_1, \dots, b_j - 1, \dots, b_q) + \epsilon_{ij}$$

is smooth if and only if $a_i + b_j = n$. (In this degeneration ϵ_{ij} is shorthand for the dimension vector $(\delta_{1i}, \dots, \delta_{pi}; \delta_{1j}, \dots, \delta_{qj}.)$

Proof. If we calculate the Euler form

$$\chi(\alpha', \epsilon_{ij}) = a_i - 1 + b_i - 1 - (n-1) = -1 + (a_i + b_i - n)$$

equals -1 if and only if $a_i + b_i = n$ $\square$

In the following we suppose that the dimension vector is ordered i.e.

$$a_1 \ge \dots \ge a_p, \ b_1 \ge \dots \ge b_q.$$

Lemma 3.12. If α is an almost simple dimension vector and $a_1 = a_2$ and $b_1 = b_2$ and $a_1 + b_1 = n$ then $a_i = b_i = 0$, $i > 2$.

Proof. We know that $\sum a_i = n$ and $\sum b_i = n$ so

$$\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j = a_1 + b_1 + a_2 + b_2 + \sum_{i>2} a_i + \sum_{j>2} b_j = 2n + \sum_{i>2} a_i + \sum_{j>2} b_j.$$

This implies that the last two terms must be zero and $a_1 = a_2 = b_1 = b_2 = \frac{n}{2}$. □

Lemma 3.13. Suppose $\alpha = (a_1, \ldots, a_p; b_1, \ldots, b_q)$ is a dimension vector of a θ-stable for which all the possible degenerations

$$(a_1, \ldots, a_i - 1, \ldots, a_p | b_1, \ldots, b_j - 1, \ldots, b_q) + \epsilon_{ij}$$

are smooth, then either

- $\alpha = (1; 1)$, wich is the trivial case;
- $b_1 = \cdots = b_q$, $a_1 > a_2$, and $a_1 + b_1 = n$ or vice versa changing the $a's$ in the $b's$.

Proof. Suppose we are not in the trivial case. If $a_1 + b_1 < n$, we can choose ϵ_{ij} randomly and $\alpha - \epsilon_{ij}$ will be simple, but by the first lemma this degeneration will not be smooth. So $a_1 + b_1 = n$.

If $a_1 = a_2$ and $b_1 = b_2$, the second lemma learns us that the dimension vector is of the form $(a, a; a, a)$ and doesn't correspond to a θ-stable.

So suppose that $a_1 > a_2$ then we will prove that all the b_j will be equal. Indeed, if this would not be the case then $b_i < b_1$. But in that case we can split off ϵ_{1i} to obtain a valid degeneration, but because $a_1 + b_i < a_1 + b_1 = n$ this degeneration will not be smooth. □

Lemma 3.14. Suppose $\alpha = (a_1, \ldots, a_p; b_1, \ldots, b_q)$, $p \leq q$ is a dimension vector of a θ-stable for which all the possible degenerations in two different simple components are smooth, then either

- $\alpha = (q - 1, 1 | 1, \ldots, 1)$;
- $\alpha = (b, b | b, b - 1, 1)$;
- $\alpha = (4, 2 | 2, 2, 2)$.

Proof. Suppose that $a_2 = a_1 - l,\ l > 0$. By Lemma 3.13 we know that all the b_i must be equal. We now distinguish the following cases

- If $l \leq q-2$ then splitting off ϵ_{1j} for $1 \leq j \leq l$ yields a term

$$(a_1 - l, a_1 - l, \ldots, a_p; \underbrace{b-1, \ldots, b-1}_{l}, b, \ldots, b)$$

  which is an almost simple dimension vector which satisfies the conditions of Lemma 3.12 so $a_3 = 0$ and $b = 1$. This gives us $\alpha = (q-1, 1; 1, \ldots, 1)$ (possibility 4).

- If $l = q-1$ then b cannot be 1 otherwise

$$a_1 + b = n \;\Rightarrow\; a_1 = q-1 \;\Rightarrow\; a_1 - l = q - 1 - (q-1) = 0$$

  which is impossible because $\theta(\alpha) = 0$. If $b \geq 2$ and a_3 is not zero, then

$$(a_1 - l, a_1 - l, a_3 - 1, \ldots, a_p; b-1, \ldots, b-1)$$

  is an almost simple dimension vector which satisfies the conditions of Lemma 3.12 so $q = 2$ and we find the solution $(b, b-1, 1; b, b)$. If $a_3 = 0$ then $2a_1 - q + 1 = qb$ and $a_1 = (q-1)b$ so $q = 3$, and because

$$(a_1 - 3, a_1 - 3; b-1, b-1, b-2)$$

  is an almost simple dimension vector which satisfies the conditions of lemma 3.2, therefore b must be two and we obtain $(4, 2; 2, 2, 2)$.

- If $q \leq l$ then we can split α in the following way:

$$\begin{aligned}\alpha =& (a_1 - q + 1, a_1 - l - 1, a_3, \ldots, a_p; b_1 - 1, \ldots, b_q - 1) \\ &+ (q-1, 1; b-1, \ldots, b-1).\end{aligned}$$

  The Euler product for this degeneration is:

$$\begin{aligned}\chi &= (a_1 - q + 1)(q-1) + (a_1 - l - 1) + q(b-1) - q^2(b-1) \\ &= (q-1)^2(b-1) + ((q-1)b - l - 1) + q(b-1) - q^2(b-1) \\ &= (q-1)^2(b-1) + (2q-1)(b-1) + q - 1 - l - 1 - q^2(b-1) \\ &= q - 1 - l - 1 < -1\end{aligned}$$

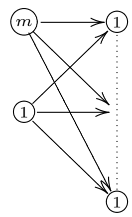
  which implies that it is not smooth.

□

3.2.3 Determining the structure of the moduli spaces

In this section we will work out the structure of the moduli spaces associated to the quivers with dimension vectors as appearing in Lemma 3.14.

First we will consider the quiver

$Q_m :=$ 

We will denote by k_i (resp. c_i) the arrow running from the first (resp. second) vertex in the left part of the bipartite quiver to the ith arrow in the right part of the quiver.

Theorem 3.15. $\mathsf{iss}_\alpha(Q_m)$ is the projective space in m dimensions.

Proof. To prove the above statement it is sufficient to show that the ring of semi-invariants is the polynomial ring in $m+1$ variables. We first prove that this ring is generated by $m+1$ semi-invariants.

All the semi-invariants are generated by the matrix-semi-invariants. Suppose that we have a representation where the arrows k_i are represented by row vectors K_i and the arrows c_i by constants C_i. A general matrix-semi-invariant of the order l is of the form

$$\begin{vmatrix} s_{11}K_1 & s_{12}C_1 & \dots & s_{1,2l-1}K_1 & s_{1,2l}C_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{m+1,1}K_{m+1} & s_{m+1,2}C_{m+1} & \dots & s_{m+1,2l-1}K_{m+1} & s_{m+1,2l}C_{m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{lm+l,1}K_{m+1} & s_{lm+l,2}C_{m+1} & \dots & s_{lm+l,2l-1}K_{lm+l} & s_{lm+l,2l}C_{m+1} \end{vmatrix},$$

where the s_{ij} represent complex numbers. Using the multilinearity in the rows, one can rewrite the big determinant as a linear combination of determinants with on each row exactly one s_{ij} equal to 1 and all the others zero.

Switching rows enables us to put them where one of the two first s's are non-zero above (mind to switch only rows modulo $m+1$). The number of such rows has to be equal to $m+1$ otherwise the determinant will be zero.

$$\left(\begin{array}{c:ccc} k \times m+1 & 0 & \cdots & 0 \\ \hdashline 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{array}\right)$$

Consequently, in the above matrix the upper left corner is a square $m+1 \times m+1$ dimensional matrix. The big deternimant now decomposes in a product of a semi-invariant of degree 1 and one of degree $l-1$. By induction all the semi-invariants are generated by the ones of degree 1. When we take a look at those we can see that by the multilineary of the determinant every such semi-invariant is a linear combination of the following one's

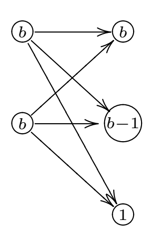
$$T_i := \begin{vmatrix} K_1 & 0 \\ \vdots & \vdots \\ K_{i-1} & 0 \\ 0 & C_i \\ K_{i+1} & 0 \\ \vdots & \vdots \\ K_{m+1} & 0 \end{vmatrix}, \ i = 1 \ldots m+1$$

Between those $m+1$ semi-invariants are no relations because if we consider an $m+1$-tuple different from zero, say $(x_1, \ldots, x_{m+1})$, the representation

$$\begin{aligned} K_i &:= \begin{pmatrix} \delta_{1i} & \cdots & \delta_{mi} \end{pmatrix}, \ 1 \leq i \leq m \\ K_{m+1} &:= \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \\ C_i &:= x_i \end{aligned}$$

has as invariant $T_i = x_i$. So we have a set of $m+1$ independent generators which make the ring of semi-invariants $\mathbb{C}[T_1 t, \ldots, T_{m+1} t]$. $\square$

Secondly, let us look at the quiver

$Q_{(b)} :=$  .

We denote the arrows between the vertices with dimension b as a and b, and the arrows from the first (resp. second) vertex of the left part of the quiver to the vertices with dimension 1 in the right part with c_1 and c_2 (resp. d_1 and d_2).

For $b = 2$ determining the moduli space is rather straightforward

Theorem 3.16. $\mathsf{iss}_\alpha(Q_{(2)})$ is the projective space in 3 dimensions.

Proof. Suppose that we have a representation where the arrows a and b are represented by 2×2-matrices A, B and the arrows c_i, d_i by row-vectors C_i, D_i. A general matrix-semi-invariant of the order l is of the form

$$\begin{vmatrix} s_{11}A & s_{12}B & \dots & s_{1,2l-1}A & s_{1,2l}B \\ s_{21}C_1 & s_{12}D_1 & \dots & s_{2,2l-1}C_1 & s_{2,2l}D_1 \\ s_{31}C_2 & s_{12}D_2 & \dots & s_{3,2l-1}C_2 & s_{3,2l}D_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{3l,1}C_2 & s_{12}D_2 & \dots & s_{2,2l-1}C_2 & s_{2,2l}D_2 \end{vmatrix},$$

where the s_{ij} represent complex numbers. Using the multilinearity in the rows, one can rewrite the big determinant as a linear combination of determinants where on each row of the C, D-part there's exactly one s_{ij} that equals 1 and all the others are zero. For the A, B-part this is not possible because they consist of two rows. But by subtracting and switching columns we can obtain a couple of rows of the form

$$\begin{pmatrix} s_1A & s_2B & 0 & \dots & 0 \end{pmatrix}.$$

Switching rows enables us to put all the rows where one of the two first s's are non-zero above (take care to switch only rows modulo 4). The number of such rows must be equal to 4, otherwise the determinant will be zero. As in the previous theorem the big determinant now decomposes in a product of a semi-invariant of degree 1 and one of degree $l-1$. By induction all the semi-invariants are generated by the ones of degree 1. When we take a look at these we can see that by the multilineary of the determinant every such semi-invariant is a linear combination of the following ones

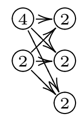
$$T_1 := \begin{vmatrix} A & 0 \\ 0 & D_1 \\ 0 & D_2 \end{vmatrix},\ T_2 := \begin{vmatrix} 0 & B \\ C_1 & 0 \\ C_2 & 0 \end{vmatrix},\ T_3 := \begin{vmatrix} A & B \\ C_1 & 0 \\ 0 & D_2 \end{vmatrix},\ T_4 := \begin{vmatrix} A & B \\ 0 & D_1 \\ C_2 & 0 \end{vmatrix}.$$

Between those 4 semi-invariants are no relations because if we consider an 4-tuple different from zero say $(x_1, \dots, x_4)$, the representation

$$\begin{array}{cc} A = \begin{pmatrix} 1 & 0 \\ 0 & x_1 \end{pmatrix} & B = \begin{pmatrix} x_2 & 0 \\ 0 & 1 \end{pmatrix} \\ C_1 = \begin{pmatrix} 0 & 1 \end{pmatrix} & C_2 = \begin{pmatrix} 0 & x_3 \end{pmatrix} \\ D_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} & D_2 = \begin{pmatrix} -x_4 & 0 \end{pmatrix} \end{array}$$

has as invariant $T_i = x_i$. So we have a set of $m+1$ independent generators and hence the ring of semi-invariants is $\mathbb{C}[T_1t, \ldots, T_4t]$. $\square$

For $b > 2$ determining the moduli space becomes more involved, but a determination of all possible degenerations shows that this moduli space is indeed smooth. The same remark holds for the quiver setting



Summarizing all the results obtained in this section we conclude with the following theorem.

Theorem 3.17. For the quiver Q the only dimension vectors for which $\mathsf{M}^{\mathsf{ss}}_{\alpha}(Q,\theta)$ is smooth are in fact

- $\alpha := (m, 1; 1, \ldots, 1)$ for which $\mathsf{M}^{\mathsf{ss}}_{\alpha}(Q,\theta) = \mathbb{P}^m$.
- $\alpha := (b, b; b, b-1, 1)$ for which $\mathsf{M}^{\mathsf{ss}}_{\alpha}(Q,\theta) = \mathbb{P}^3$ if $b = 2$.
- $\alpha := (4, 2; 2, 2, 2)$.

3.3 Other free products

3.3.1 More Factors

Using the same tools as in the sections before, we can look at some more complicated constructions of free products and their representation spaces. First of all we could try to generalize to free products of more than two cyclic groups. Suppose we want to look at the representations of the group

$$\mathbb{Z}_{p_1} * \mathbb{Z}_{p_2} * \cdots * \mathbb{Z}_{p_k}.$$

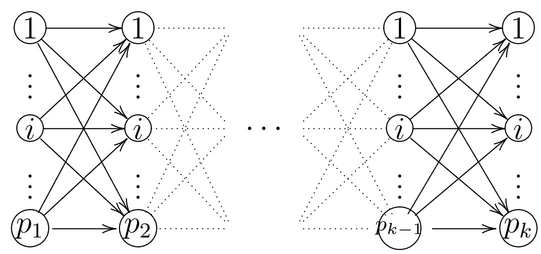
For every n-dimensional representation we can split up $\mathbb{C}^n$ in k different ways as a direct sum of eigenspaces $\oplus_i V_i^l, l = 1, \ldots, k$ and for every couple $(s,t), 1 \leq s,t \leq k$ we have maps m_{ij}^{st} according to the diagram

$$\begin{tikzcd}
V_1^s \oplus \cdots \oplus V_{p_s}^s \arrow[r, "\cong"] & \mathbb{C}^n \arrow[r, "\cong"] & V_1^t \oplus \cdots \oplus V_{p_t}^t \arrow[d] \\
V_i^s \arrow[u, hook] \arrow[rr, "m_{ij}^{st}"'] & & V_j^t
\end{tikzcd}$$

adding up to invertible matrices M^{st}. These invertible matrices satisfy the following relations

$$\forall s, t, u \leq k : M^{st}M^{tu} = M^{su}.$$

So it suffices to know the subset of matrices $M^{12}, M^{23}, \ldots, M^{k-1k}$ in order to know the representation. Between those set of matrices there exist no further relations. Transferring this to the quiver language every representation of our group is equivalent to a representation of the quiver



Again, not every quiver representation gives us a group representation because the big matrices M^{st} must be invertible. The difference between the free product of two groups and more groups, is that more than one matrix must be invertible. Therefore the representation space of our group can be seen as the representation space of the algebra

$$\mathbb{C}Q_{\mathcal{M}^{12},\ldots,\mathcal{M}^{k-1k}}.$$

which is the universal localization of $\mathbb{C}Q$ to the set of $k-1$-matrices which are similar to the matrix $\mathcal{M}$ in the previous section.

The method we use is the same as for the free product of 2 abelian groups. First of all we have a collection of very elementary simple representations corresponding to a dimension vector $\epsilon_{i_1 \ldots i_k}$ which has dimension 1 on the i_j^{th} vertex of the j^{th} column and zero every where else. Every vertex of with a dimension that satisfies the condition that the dimension of every column vertices is the same can be decomposed as a sum of dimension vectors of the ϵ-form. Therefore the non-empty representation spaces are exactly the one with dimension vectors that satisfy this condition.

To check whether there are simples of these dimension vectors, we have to look at the local quiver coming from a decomposition in elementary simples.

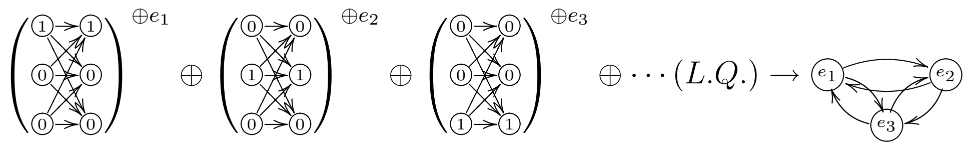
To look for quotient spaces $\mathsf{iss}_\alpha \mathbb{Z}_{p_1} * \cdots * \mathbb{Z}_{p_k}$ that are smooth we look at certain decompositions. It is however not necessary to know as dimension vectors of simples to solve the smoothness question because as we will see, using the elementary simples, we can show that there are new smooth spaces.

Theorem 3.18. Every representation space with dimension vector α for which there are 3 columns with at least 2 vertices of non-zero dimension or 2 columns with 3 non-zero vertices, contains singularities.

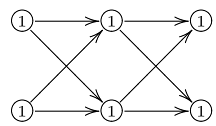
Proof. We observe that

$$\chi_Q(\epsilon_{i_1\dots i_k}, \epsilon_{j_1\dots j_k}) = \delta_{i_1 j_1} + \cdots + \delta_{i_k j_k} - k + 1$$

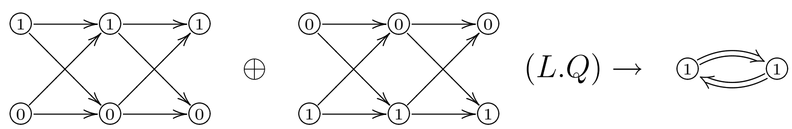
So whenever two $\epsilon' s$ differ at two places in the local quiver there is an arrow between them. If there are 2 columns with 3 vertices we can make a decomposition like



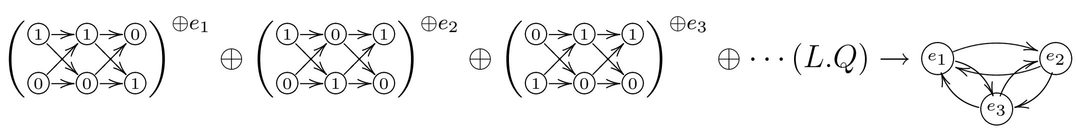
As soon as there are 3 column with 2 vertices of nonzero dimension, we either are in the case where $n = 2$ and we have the situation



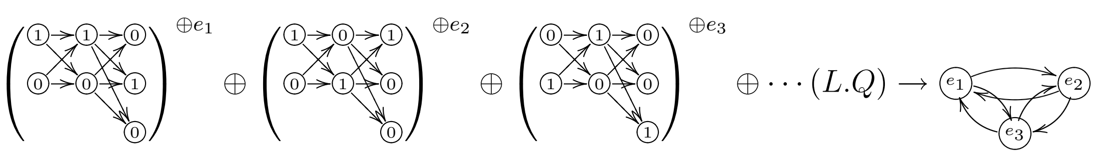
Therefore we can construct a decomposition like



If $n > 2$ and in each column there are exactly 2 nonzero vertices, we can suppose that the dimensions of the upper vertex in each column is bigger than 1. In this case we can make a decomposition like



Finally the last situation is that one of the 3 columns has more than 3 nonzero vertices. This gives us a decomposition like



$\square$

So if $\mathsf{iss}_\alpha \mathbb{Z}_{p_1} * \cdots * \mathbb{Z}_{p_k}$ is a smooth space this means that for almost all but 2 factors the representation must reduce to a single eigenspace, f.i. $\mathbb{Z}_{p_i}$ and $\mathbb{Z}_{p_j}$. In such case it is easy to check that the corresponding quotient space is the same as $\mathsf{iss}_{\alpha'} \mathbb{Z}_{p_i} * \cdots * \mathbb{Z}_{p_j}$ where α' is the dimension vector α reduced to those two relevant components. So we can conclude that nothing new happens in the case of more factors.

3.3.2 General finite groups

Another way to extend the problem is to look at general finite groups that are not necessarily commutative. As we know a finite group is always reductive so every finite dimensional representation is semisimple. Moreover every finite group has only a finite collection of non-isomorphic simple representations.

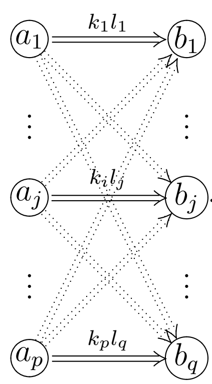
Suppose that G and H are two finite groups and that $\rho : \mathsf{G} * \mathsf{H} \to \mathsf{GL}(V)$ is a representation of their free product. Because of the reducibility, we can again decompose V in 2 ways as a direct sum of representations of G and H.

$$V = \underbrace{V_1^{\oplus a_1} \oplus \cdots \oplus V_p^{\oplus a_p}}_{\text{as a } \mathsf{G}\text{-representation}} = \underbrace{W_1^{\oplus b_1} \oplus \cdots \oplus W_q^{\oplus b_q}}_{\text{as a } \mathsf{H}\text{-representation}}$$

Here V_i is a simple G representation of dimension k_i for wich the set of vectors $\{v_i^1, \dots, v_i^{k_i}\}$ forms a basis, W_j is a simple H representation of dimension l_j with basis $\{w_j^1, \dots, w_j^{l_j}\}$. For every quadruple (i, j, s, t), $1 \le i \le p, 1 \le j \le q, 1 \le s \le k_i, 1 \le t \le l_j$ we can draw the following diagram

$$\begin{tikzcd}
V_i^{\oplus a_i} \arrow[r, hook] & V_1^{\oplus a_1} \oplus \cdots \oplus V_p^{\oplus a_p} = V = W_1^{\oplus b_1} \oplus \cdots \oplus W_q^{\oplus b_q} \arrow[r, two heads] & W_j^{\oplus b_j} \arrow[d, "\pi_{\mathbb{C}^{b_j} \otimes w_j^t}"] \\
\mathbb{C}^{a_i} \arrow[u, "\otimes v_i^s"] \arrow[rr, "m_{ij}^{st}"] & & \mathbb{C}^{b_j}
\end{tikzcd}$$

So we have a collection of $p + q$ complex vector spaces with between each couple of spaces $\mathbb{C}^{a_i}$, $\mathbb{C}^{b_j}$ $k_i l_j$ linear maps m_{ij}^{st}. In fact this is a representation of the quiver setting

$(Q, \alpha) :=$ 

.

Because the maps m_{ij}^{st} must add up to an invertible map, we have the equation

$$a_1 k_1 + \cdots + a_p k_p = b_1 l_1 + \cdots + b_q l_q.$$

This implies also that we can see $\mathsf{Rep}_\alpha \mathsf{G} * \mathsf{H}$ can be seen as a subspace of the semistable (Q, α)-representations according to the character

$$\theta := (-k_1, \dots, -k_p; l_1, \dots, l_q).$$

The last thing that we have to check is that the actions on both spaces are the same.

On the representation V acts its automorphism group as a G-representation $\mathsf{Aut}_{\mathsf{G}}V$ and its automorphism group as a H-representation $\mathsf{Aut}_{\mathsf{H}}V$. Applying automorphisms of both groups won't change the structure of the representation, so the group acting on $\mathsf{Rep}_\alpha \mathsf{G} * \mathsf{H}$ is $\mathsf{Aut}_{\mathsf{G}}V \oplus \mathsf{Aut}_{\mathsf{H}}V$. As we know from the first chapter

$$\mathsf{Aut}_{\mathsf{G}}V \oplus \mathsf{Aut}_{\mathsf{H}}V = (\mathsf{GL}_{a_1}(\mathbb{C}) \oplus \cdots \oplus \mathsf{GL}_{a_p}(\mathbb{C})) \oplus (\mathsf{GL}_{b_1}(\mathbb{C}) \oplus \cdots \oplus \mathsf{GL}_{b_q}(\mathbb{C})) = \mathsf{GL}_\alpha,$$

and one can easily check that the action on the maps m_{ij}^{st} is the same as one expects in the quiver situation. Because the m_{ij}^{st} must add up to an invertible map we can again see $\mathsf{Rep}_\alpha \mathsf{G} * \mathsf{H}$ as the representation space $\mathsf{Rep}_\alpha \mathbb{C}Q_{\mathcal{M}}$ where $\mathcal{M}$ is the matrix with as entries all possible arrows of the quiver. We can conclude with the following diagram

$$\begin{tikzcd}
\mathsf{Rep}_\alpha \mathsf{G} * \mathsf{H} = \mathsf{Rep}_\alpha \mathbb{C}Q_{\mathcal{M}} \arrow[r, hook] \arrow[d] & \mathsf{Rep}_\alpha^{ss}(Q,\theta) \arrow[d] \\
\mathsf{iss}_\alpha \mathsf{G} * \mathsf{H} = \mathsf{iss}_\alpha \mathbb{C}Q_{\mathcal{M}} \arrow[r, hook] & \mathsf{M}_\alpha^{\mathsf{ss}}(Q,\theta)
\end{tikzcd}$$

In order to find the smooth moduli spaces we first have to find all possible dimension vectors for which θ-stable representations exists. This is a far from trivial question, first of all we cannot decompose every solution to $\theta \cdot \alpha = 0$ as a sum of dimension vectors of simples we already know like in the case of $\mathbb{Z}_p \times \mathbb{Z}_q$. Therefore one must check for all solutions that cannot be written as a sum of smaller solutions whether there exist simples. This cannot be done using techniques of local quivers and is hence beyond the scope of this thesis.

Even without knowing the exact dimension vectors of simples we can say some things about the smoothness of certain moduli spaces. Let Σ_θ denote the set of dimension vectors for which there exist θ-stable representation. On this set we have the natural partial ordering, $<$.

If α is a minimal element in Σ_θ according to this ordering, this means that every semi-stable α-dimensional representation in $\mathsf{Rep}_\alpha^{ss} Q$ is in fact stable. Every semi-stable representation of dimension vector α corresponds to a representation of a certain localization of $\mathbb{C}Q_{\mathcal{N}}$. Looking at its Jordan-Hölder decomposition it can only have one factor because α is a minimal vector for which there exist θ-stables and hence simples. Therefore this $\mathbb{C}Q_{\mathcal{N}}$-representation is simple and the corresponding semi-stable representation is simple. Looking at the local quiver setting of a simple representation it has one vertex of dimension one and several loops. This quiver setting is thus coregular and we can conclude that

Lemma 3.19. If α is a minimal element of Σ_θ then $\mathsf{M}^{\mathsf{ss}}_\alpha(Q,\theta)$ is a smooth space.

It would be a nice result that these would be the only dimension vectors for which $\mathsf{M}^{\mathsf{ss}}_\alpha(Q,\theta)$ is smooth. In general this is not true, for instance if we look at the case of $\mathbb{Z}_p * \mathbb{Z}_q$ this is definitely wrong (the mininal vectors here are of dimension ϵ_{ij}). However if we restrict to a special case of θ we can prove such a theorem.

Theorem 3.20. If θ does not contain any ± 1, $\mathsf{M}^{\mathsf{ss}}_\alpha(Q,\theta)$ is a smooth space if and only if α is a minimal element of Σ_θ.

Proof. If we look at two dimension vectors $\alpha, \alpha' \in \Sigma_\theta$, where $\alpha = (a_1, \ldots, a_p; b_1, \ldots, b_q)$ and α' is similar, we see that

$$\begin{aligned}
\chi_Q(\alpha,\beta) &= \sum_i a_i a_i' + \sum_i b_i b_i' - \sum_{i,j} a_i k_i l_j b_j' \\
&= \sum_i a_i a_i' + \sum_i b_i b_i' - \frac{\sum_{i,j} a_i k_i k_j a_j' + b_i l_i l_j b_j'}{2} \\
&\leq \sum_i a_i a_i' - 2(\sum_i a_i)(\sum_i a_i') + \sum_i b_i b_i' - 2(\sum_i b_i)(\sum_i b_i') \\
&\leq -(\sum_i a_i)(\sum_i a_i') - (\sum_i b_i)(\sum_i b_i') \\
&\leq -2
\end{aligned}$$

If we take $\alpha = \alpha'$ we see the dimension of $\mathsf{M}^{\mathsf{ss}}_\alpha(Q,\theta)$ is nonzero, so for every $\alpha \in \Sigma_\theta$ there exists an infinite number of simples.

If α is not a minimal dimension vector we can find a decomposition of α containing at least 2 different simples with multiplicity 1. The number of arrows between them in the local quiver is at least 2 so we have a subquiver of the form $\textcircled{1} \rightleftarrows \textcircled{1}$. $\square$

Chapter 4

Symplectic Geometry and Preprojective algebras

In this chapter we look at some algebraic structures that are closely related to the quiver algebra and the varieties $\mathsf{Rep}_\alpha Q$ and $\mathsf{iss}_\alpha Q$. Using the structure theorem for symmetric quivers we will describe explicitely the simplest cases for such algebra's. The subjects and results treated here mostly come from [5], and are independently found by Victor Ginzburg in [14].

4.1 Introduction

Consider a vector space V of dimension n. Its cotangent space is $V \oplus V^*$. We can identify $\mathbb{C}[V \oplus V^*]$ with the polynomial ring $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ where x_i are the coordinate functions from the basis of V, and the y_i come from the corresponding duals.

From symplectic geometry we rec all that a cotangent bundle admits a canonical symplectic structure. In the case of T^*V, this structure $\omega \in \Omega^2(\mathrm{T}^*V)$ has the following form.

$$\omega := \sum_{i=1}^{n} dx_i \wedge dy_i$$

where $x_i, y_i \in \mathbb{C}[V \oplus V^*]$ are the coordinate functions in the ring $\mathbb{C}[V \oplus V^*]$. This symplectic structure gives us an identification between vectorfields and 1-forms in the following way

$$\tau : \mathsf{Vect}(V \oplus V^*) \to \Omega^1(V \oplus V^*) : X \mapsto i_X\omega := \omega(X, \cdot)$$

Using this identification we can construct a map from the polynomial functions over $V \oplus V^*$ to the vector fields

$$\mathbb{C}[V \oplus V^*] \xrightarrow{d} \Omega^1(V \oplus V^*) \xrightarrow{\tau^{-1}} \mathsf{Vect}(V \oplus V^*)$$

The image of this map is the set of symplectic vector fields, i.e.

$$\mathsf{Vect}_\omega(V \oplus V^*) := \{X \in \mathsf{Vect}(V \oplus V^*) | \mathcal{L}_X \omega = 0\}$$

where $\mathcal{L}_X = d \circ i_X + i_X \circ d$ is the Lie -derivation associated to X. $\mathsf{Vect}_\omega(V \oplus V^*)$ forms a Lie algebra under the normal commutator and we can make the map $\tau^{-1}d$ into a Lie algebra morphism by defining a new bracket operation on $\mathbb{C}[V \oplus V^*]$.

Definition 4.1. Define the Poisson bracket $\{,\}$ on $\mathbb{C}[V \oplus V^*]$ as

$$\{f, g\} = \sum_i \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}.$$

This bracket makes $\mathbb{C}[V \oplus V^*]$ into a Poisson algebra (a commutative algebra equiped with a Lie bracket such that the Leibniz law, $[a, bc] = b[a, c] + [a, b]c$ holds)

If G acts on V and V^*, this group action leaves the symplectic structure on $V \oplus V^*$ invariant. This implies that the subring of invariant functions over $V \oplus V^*$ closed is under the Poisson bracket and hence a Poisson algebra as well.

Because G acts symplectic the Lie algebra $\mathfrak{g}$ will be mapped by the action to the symplectic vectorfields.

$$\mathfrak{g} \to \mathsf{Vect}_\omega(V \oplus V^*) : g \mapsto X_g : X_g(v, *v) = (gv, gv^*)$$

In this case we can do even more, we can construct a map H form $\mathfrak{g}$ to the polynomial functions such that we have the following commutative diagram.

$$\begin{tikzcd} & \mathfrak{g} \arrow[dl, dotted, "H"'] \arrow[d] \\ \mathbb{C}[V \oplus V^*] \arrow[r, "\tau^{-1}d"'] & \mathsf{Vect}_\omega(V \oplus V^*). \end{tikzcd}$$

In this diagram the map H is called the Hamiltonian and looks like

$$H : \mathfrak{g} \to \mathbb{C}[V \oplus V^*] : g \mapsto f_g : f_g(v, v^*) = v(gv^*)$$

Dualizing this map we create a new map μ form $V \oplus V^*$ to the dual of the Lie algebra $\mathfrak{g}$

$$\mu : V \oplus V^* \to \mathfrak{g}^* \text{ s.t. } \forall v \in V \oplus V^*, \forall g \in \mathfrak{g} : \mu(v)(g) := H(g)(v)$$

This map is called the moment map of the G-action. This map is an important tool in symplectic geometry and as we will see in what follows that it is also usefull in representation theory.

In the next sections we will study all the invariant theoretic and geometric notions we introduced here in the case of representation spaces for quivers and connect them to concepts from non-commutative geometry.

4.2 Necklaces

Let Q be a quiver. Recall that a cycle is a path in the quiver with the same starting end terminating vertex. We call the set of all cyclic permutations of a cycle a *necklace word*. The set of all necklace words of a quiver will be denoted by Neckl. Remark that we also include the paths of length zero in this set.

If we have a dimension vector α we define for a necklace word w, $Tr_\alpha w$ to be the corresponding invariant function by taking the trace of the cycle. Theorem 2.1 can now be restated as

Theorem 4.2. $\mathsf{Tr}_\alpha : \mathbb{C}[\mathsf{Neckl}Q] \to \mathbb{C}[\mathsf{iss}_\alpha Q]$ is surjective, where $\mathbb{C}[\mathsf{Neckl}Q]$ is the polynomial algebra with as variables the necklace words and Tr_α is the algebra morphism induced by $w \mapsto \mathsf{Tr}_\alpha w$.

Suppose we have a quiver setting (Q, α) and we look at the dual space of $\mathsf{Rep}_\alpha Q$:

$$\mathsf{Rep}_\alpha Q^* = \bigoplus_{a \in A} \mathsf{Mat}_{\alpha_{t(a)} \times \alpha_{s(a)}}(\mathbb{C})^* = \bigoplus_{a \in A} \mathsf{Mat}_{\alpha_{s(a)} \times \alpha_{t(a)}}(\mathbb{C}).$$

This vector space can be seen as the representation space, $\mathsf{Rep}_\alpha Q^{op}$, of the opposite quiver Q^{op} which has the same vertices as Q but all arrows reversed.

Apart from the dual space of $\mathsf{Rep}_\alpha Q$ we can also look at the cotangent space of this variety. Because $\mathsf{Rep}_\alpha Q$ is a vector space this cotangent space is of the form

$$\mathrm{T}^*\mathsf{Rep}_\alpha Q = \mathsf{Rep}_\alpha Q \oplus \mathsf{Rep}_\alpha Q^* = \mathsf{Rep}_\alpha Q \oplus \mathsf{Rep}_\alpha Q^{op},$$

and can again be seen as the representation space, $\mathsf{Rep}_\alpha Q^d$, of the double quiver which has the same vertices and arrows as Q but for each arrow $a \in Q$, Q^d has an extra arrow in the opposite direction which we will denote by a^*.

All these identifications are not only correct from the pure geometric point of view, but also if we look at the action of GL_α.

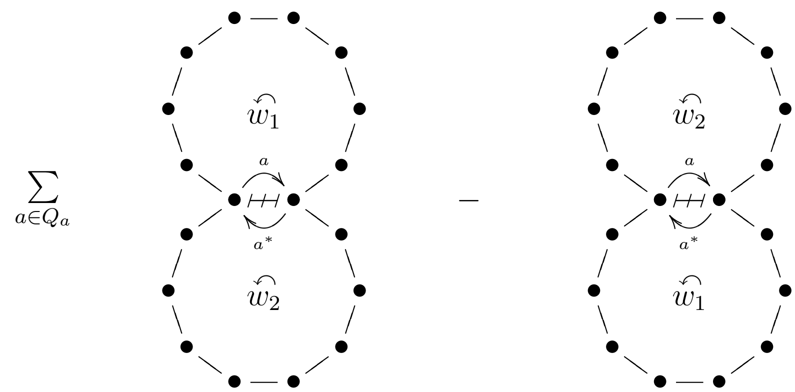


Figure 4.1: Lie bracket $[w_1, w_2]$ in $\mathfrak{Lie}_Q$.

If we now want to look at the structure of the Poisson algebra arising from the symplectic structure on $\mathrm{T}^*\mathsf{Rep}_\alpha Q$, we can identify this structure as a vector space with $\mathbb{C}[\mathsf{iss}_\alpha Q^d]$. So in fact for every dimension vector α we have to construct a Poisson bracket, on $\mathbb{C}[\mathsf{iss}_\alpha Q^d]$ giving it his additional structure.

A more appealling way to look at this problem is to construct a new bracket on the big ring $\mathbb{C}[\mathsf{Neckl} Q^d]$ and then transport this structure for every α to $\mathbb{C}[\mathsf{iss}_\alpha Q^d]$ via the trace map Tr_α. This can indeed be done.

Definition 4.3. The *necklace Lie algebra* $\mathfrak{Lie}_Q$ for the quiver Q has as basis the set of all necklace words w for the *double* quiver Q^d and where the Lie bracket $[w_1, w_2]$ is determined as in figure 4.1.

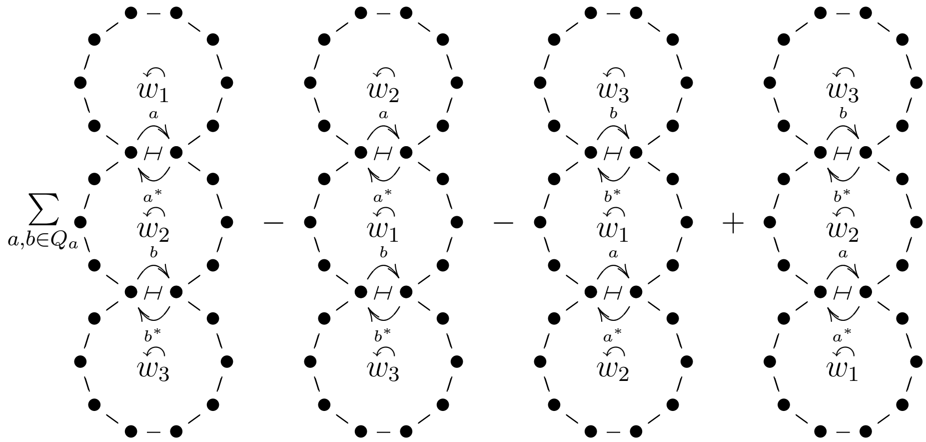
That is, for every arrow $a \in Q_a$ we look for an occurrence of a in w_1 and of a^* in w_2. We then open up the necklaces by removing these factors and regluing the open ends together to form a new necklace word.

We repeat this operation for *all* occurrences of a (in w_1) and a^* (in w_2). We then replace the roles of a^* and a and redo this operation with a minus sign. Finally, we add up all these obtained necklace words for all arrows $a \in Q_a$.

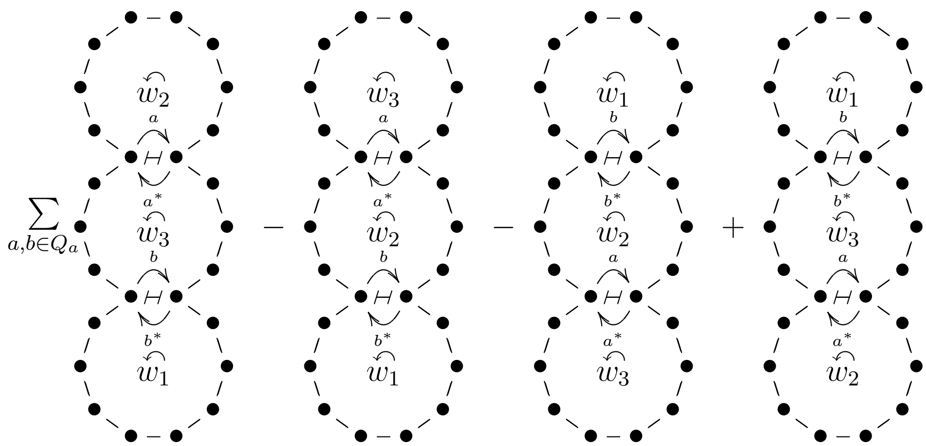
Using this graphical description the Jacobi identity for $\mathfrak{Lie}Q$ follows from figure 4.2. The *Necklace Poisson algebra*,$\mathfrak{Poiss}Q$,is the Poisson structure on $\mathbb{C}[\mathsf{Neckl} Q^d]$ coming from the Lie algebra structure on the subspace $\mathfrak{Lie}Q$ spanned by the generators.

Theorem 4.4. $\mathsf{Tr}_\alpha : \mathfrak{Poiss}Q \to \mathbb{C}[\mathsf{iss}_\alpha Q]$ is a Poisson algebra morphism.

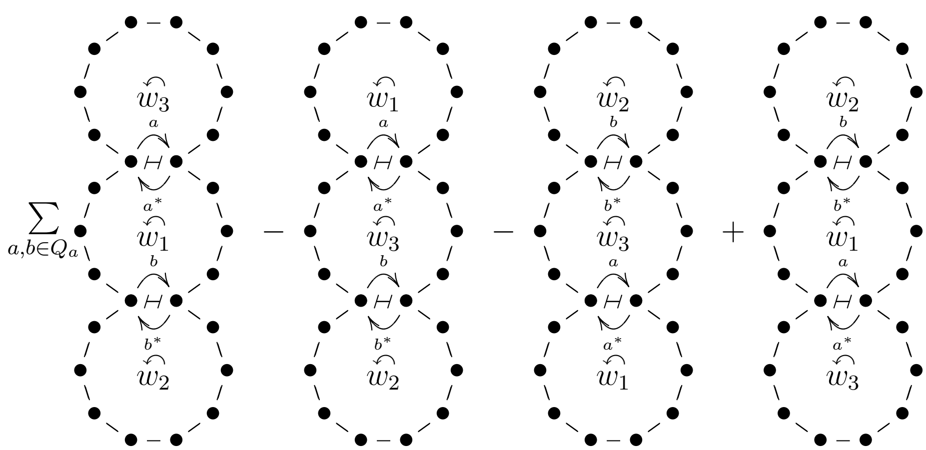
Proof. In the case of quiver representations it is easy to check that this symplectic structure is obtained by trace pairing. For every arrow $\textcircled{i} \xleftarrow{a} \textcircled{j}$ in Q we have an $\alpha_i \times \alpha_j$-dimensional matrix of coordinate functions X^a. For the adjoined arrow



$[[w_1, w_2], w_3]$



$[[w_2, w_3], w_1]$



$[[w_3, w_1], w_2]$

Figure 4.2: Jacobi identity for the necklace Lie algebra $\mathfrak{Lie}_Q$. Term $1a$ vanishes against $2c$, term $1b$ against $3d$, $1c$ against $3a$, $1d$ against $2b$, $2a$ against $3c$ and $2d$ against $3b$.

a^* we denote the $\alpha_j \times \alpha_i$-dimensional matrix of coordinate functions X^{a^*}. We can write down our closed 2-form as

$$\omega = \sum_{a \in A} \mathsf{Tr}(dX^a \wedge dX^{a^*}).$$

Where d works on the entries of the matrices. This symplectic structure induces a Poisson Bracket on the coordinate ring $\mathbb{C}[\mathsf{Rep}_\alpha Q^d]$. If we use the convention that ∂_a stands for the matrix of differential operators

$$\begin{pmatrix} \frac{\partial}{\partial f^a_{11}} & \cdots & \frac{\partial}{\partial f^a_{1\alpha_j}} \\ \vdots & & \vdots \\ \frac{\partial}{\partial f^a_{\alpha_i 1}} & \cdots & \frac{\partial}{\partial f^a_{\alpha_I \alpha_j}} \end{pmatrix}$$

and a similar convention for ∂_{a^*}, we can write down the bracket explicitely:

$$\{f, g\} = \sum_{a \in A} (\mathsf{Tr} \partial_a f \partial_{a^*} g - \mathsf{Tr} \partial_{a^*} f \partial_a g)$$

Suppose now that f and g are traces of the necklace words $w_1 \sim p_1 \cdots p_l$ and $w_2 \sim q_1 \cdots q_m$, where $p_i, q_i \in A$ then

$$\partial_a f = \sum_{i=1}^{l} \delta_{p_i a} X^{p_{i+1}} \cdots X^{p_l} X^{p_1} \cdots X^{p_{i-1}}$$

is equal to the sum of the representation matrices of paths obtained by opening w_1 and removing one occurance of a. Working out the product with the matrix $\partial_{a^*} g$ and taking the trace, we can identifie $\mathsf{Tr} \partial_a f \partial_{a^*} g$ with the sum of traces of all paths obtained by deleting an a in w_1 and a a^* in w_2 and connecting the two chains. This is exactly what happens in the definition of the bracket on $\mathfrak{Poiss} Q$ so we can conclude that Tr_α is a Poisson algebra morphism. □

We now use this theorem to determine the Poisson algebras that occur in the simplest case when Q^d is coregular.

Theorem 4.5. The Poisson algebras $\mathfrak{Poiss}_\alpha Q$ for which the underlying ring is polynomial come from quivers that are connected sums over ones of the following settings:

- $(Q, \alpha) = \textcircled{1}$ for which $\mathfrak{Poiss}_\alpha Q = \mathbb{C}[X, Y], \{X, Y\} = 1$
- $(Q, \alpha) = \textcircled{2}$ for which $\mathfrak{Poiss}_\alpha Q = \mathbb{C}[X_1, X_2, Y_1, Y_2, Z]$ with bracket

$$\begin{aligned} &\{X_i, X_j\} = \{Y_i, Y_j\} = 0 \\ &\{X_1, Y_1\} = 1 \\ &\{X_2, Y_1\} = \{X_1, Z\} = X_1 \\ &\{X_1, Y_2\} = \{Z, Y_1\} = Y_1 \\ &\{X_2, Y_2\} = \{X_2, Z\} = \{Y_2, Z\} = Z \end{aligned}$$

- $(Q,\alpha) = 1 \longleftrightarrow n \longleftrightarrow m$ for which the bracket is trivial.

- $(Q,\alpha) = n \longleftrightarrow 2 \longleftrightarrow m$ for which the bracket is trivial.

- $(Q,\alpha) = 1 \overset{k}{\Longleftrightarrow} n$ $n \geq k$ for which $\mathfrak{Poiss}_\alpha Q = \mathbb{C}[X_{ij}, 1 \leq i,j \leq k]$ with bracket
$$\{X_{ij}, X_{kl}\} = \delta_{il}X_{jk} - \delta_{jk}X_{il}.$$

The Poisson algebra of a connected sum of these settings is the tensor product of the corresponding Poisson algebras.

Proof. If $\mathfrak{Poiss}_\alpha Q$ its underlying algebra is polynomial, the double quiver setting (Q^d, α) must be coregular and hence Q must be the connected sum of quivers settings whose double is one of the settings in theorem 2.20 or a vertex with dimension 1 or 2 and 2 loops. Moreover it is easy to see that the Poisson algebra the connected sum of two such settings will be the tensor product of the 2 Poisson algebras because the Poisson brackets of cycles belonging to different components is zero. To finish the proof we have to make a list of all the possible components of which the double is coregular and calculate their Poisson algebra.

- $(Q,\alpha) = \overset{\circlearrowleft}{1}$. $\mathbb{C}[\mathsf{iss}_\alpha Q^d] = \mathbb{C}[\mathsf{Rep}_\alpha Q^d]$ is generated by the two loops and their bracket is 1.

- $(Q,\alpha) = 2 \circlearrowright \ell$. $\mathbb{C}[\mathsf{iss}_\alpha Q^d]$ is generated by five generators $X_i := \mathsf{Tr}V_\ell^i, Y_i := \mathsf{Tr}V_{\ell^*}^i, Z := \mathsf{Tr}V_\ell V_{\ell^*}$. The Poisson brackets can be easily checked using the graphic formula.

- In $(Q,\alpha) = 1 \longleftrightarrow n \longleftrightarrow m$ and $(Q,\alpha) = n \longleftrightarrow 2 \longleftrightarrow m$, every generator comes from a cycle that is composed of subcycles of the form aa^* where a is an arrow of Q. Therefore the two sums in the commutator bracket are identical so the bracket is trivial.

- Suppose $(Q,\alpha) = 1 \overset{k}{\Longleftrightarrow} n$ $n \geq k$ then $\mathsf{iss}_\alpha Q^d$ is generated by cycles containing one arrow to the left and one arrow to the right. If a points to the left, then a^* points of course to the right and vice versa. Now define for each couple of 2 non-starred arrows $a, b \in A$ the generator X_{ab} which is the trace of the cycle containing either a of a^* going to the left and b or b^* going to the right, multiplicated by i if you used a^* and once more multiplicated by i if you used b.

  If we now compute the bracket of two such generators we see that
$$\{X_{ab}, X_{cd}\} = \delta_{ad}X_{cb} - \delta_{cb}X_{ad}.$$

because the only two cycles we can form out of this are obtained according to the arrows a and d or c and b (with the obvious stars). To obtain the first cycle d must the starred version of a or the the other way round. If we are in the first case the cycle comes from the first part of the bracket expression and we have a plus sign. In the second case we would have a minus sign but the two i-factors we put in the definition of the generators $X_{...}$ turn the minus sign we obtain from the bracket expression again into a plus sign, therefore the expression above holds for all $X's$.

□

4.3 Non-commutative Differential Forms

In this section we will introduce non-commutative differential forms over the path algebra of the quiver. We will compute the deRham cohomology of this algebra, and prove that it is zero except in degree zero. This fact is a generatization of Kontsevich's acyclicity result for the free algebra that can be found in [17]. Using this fact together with a non-commutative symplectic structure we will give a Poisson interpretation to the Lie bracket on $\mathfrak{Lie}Q$.

The crucial idea is to consider the *relative* differential forms (as defined in [8]) of $\mathbb{C}Q$ with respect to the semisimple subalgebra $\mathbb{C}V = \mathbb{C} \times \ldots \times \mathbb{C}$ generated by the vertex idempotents, $v_1, \ldots, v_k \in V$. The idea being that in considering quiver representations one works in the category of $\mathbb{C}V$-algebras rather than $\mathbb{C}$-algebras.

For a subalgebra B of A, let $\overline{A}_B$ denote the cokernel of the inclusion as B-bimodule. The space of relative differential forms of degree n of A with respect to B is

$$\Omega^n_B\ A = A \otimes_B \underbrace{\overline{A}_B \otimes_B \ldots \otimes_B \overline{A}_B}_{n}$$

The space $\Omega^\bullet_B\ A$ is given a differential graded algebra structure by taking the multiplication

$$(a_0, \ldots, a_n)(a_{n+1}, \ldots, a_m) = \sum_{i=0}^{n} (-1)^{n-i}(a_0, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_m)$$

and the differential $d(a_0, \ldots, a_n) = (1, a_0, \ldots, a_n)$, see [8]. Here, $(a_0, \ldots, a_n)$ is a representant of the class $a_0 da_1 \ldots da_n \in \Omega^n_B\ A$ and we recall that $\Omega^\bullet_B\ A$ s generated by the a and da for all $a \in A$. The *relative cohomology* $\mathsf{H}^n_B\ A$ is defined as the cohomology of the complex $\Omega^\bullet_B\ A$.

For $\theta \in \mathsf{Der}_B\ A$, the Lie algebra of B-derivations of A (that is θ is a derivation of A and $\theta(B) = 0$), we define a degree preserving derivation $\mathcal{L}_\theta$ and a degree -1 super-derivation i_θ on $\Omega^\bullet_B\ A$ (that is, for all $\omega \in \Omega^i_B\ A$ we have that $i_\theta(\omega\omega') = i_\theta(\omega)\omega' + (-1)^i \omega i_\theta(\omega')$)

$$\begin{tikzcd}
\Omega^{n-1}_B\ A \arrow[r, bend left, "d"] \arrow[loop below, "\mathcal{L}_\theta"] & \Omega^n_B\ A \arrow[l, bend left, "i_\theta"] \arrow[r, bend left, "d"] \arrow[loop below, "\mathcal{L}_\theta"] & \Omega^{n+1}_B\ A \arrow[l, bend left, "i_\theta"] \arrow[loop below, "\mathcal{L}_\theta"]
\end{tikzcd}$$

by the rules

$$\begin{cases} \mathcal{L}_\theta(a) = \theta(a) & \mathcal{L}_\theta(da) = d\ \theta(a) \\ i_\theta(a) = 0 & i_\theta(da) = \theta(a) \end{cases}$$

for all $a \in A$. We have the Cartan homotopy formula $\mathcal{L}_\theta = i_\theta \circ d + d \circ i_\theta$ as both sides are degree preserving derivations on $\Omega^\bullet_B\ A$ and they agree on all the generators a and da for $a \in A$.

Lemma 4.6. Let $\theta, \gamma \in \mathsf{Der}_B\ A$, then we have on $\Omega^\bullet_B\ A$ the identities of operators

$$\begin{cases} \mathcal{L}_\theta \circ i_\gamma - i_\gamma \circ \mathcal{L}_\theta = [\mathcal{L}_\theta, i_\gamma] & = i_{[\theta,\gamma]} = i_{\theta\circ\gamma-\gamma\circ\theta} \\ \mathcal{L}_\theta \circ \mathcal{L}_\gamma - \mathcal{L}_\gamma \circ \mathcal{L}_\theta = [\mathcal{L}_\theta, \mathcal{L}_\gamma] & = \mathcal{L}_{[\theta,\gamma]} = \mathcal{L}_{\theta\circ\gamma-\gamma\circ\theta} \end{cases}$$

Proof. Consider the first identity. By definition both sides are degree -1 super-derivations on $\Omega^\bullet_B\ A$ so it suffices to check that they agree on generators. Clearly, both sides give 0 when evaluated on $a \in A$ and for da we have

$$(\mathcal{L}_\theta \circ i_\gamma - i_\gamma \circ \mathcal{L}_\theta)da = \mathcal{L}_\theta\ \gamma(a) - i_\gamma\ d\ \theta(a) = \theta\ \gamma(a) - \gamma\ \theta(a) = i_{[\theta,\gamma]}(da)$$

A similar argument proves the second identity. □

We now specialize to the quiver-case with $A = \mathbb{C}Q$ the path algebra and $B = \mathbb{C}V = \mathbb{C}^k$ the vertex algebra.

Lemma 4.7. Let Q be a quiver on k vertices, then a basis for $\Omega^n_{\mathbb{C}V}\ \mathbb{C}Q$ is given by the elements

$$p_0 dp_1 \dots dp_n$$

where p_i is an oriented path in the quiver such that *length* $p_0 \geq 0$ and *length* $p_i \geq 1$ for $1 \leq i \leq n$ and such that the starting point of p_i is the endpoint of p_{i+1} for all $1 \leq i \leq n-1$.

Proof. Clearly $l(p_i) \geq 1$ when $i \geq 1$ or p_i would be a vertex-idempotent whence in V. Let v be the starting point of p_i and w the end point of p_{i+1} and assume that $v \neq w$, then

$$p_i \otimes_{\mathbb{C}V} p_{i+1} = p_i v \otimes_{\mathbb{C}V} w p_{i+1} = p_i v w \otimes_{\mathbb{C}V} p_{i+1} = 0$$

from which the assertion follows. □

Theorem 4.8. Let Q be a quiver on k vertices, then the relative differential form-complex has the following cohomology

$$\begin{cases} \mathsf{H}^0_{\mathbb{C}V}\ \mathbb{C}Q & \simeq \mathbb{C}\times\ldots\times\mathbb{C}\ (k \text{ factors}) \\ \mathsf{H}^n_{\mathbb{C}V}\ \mathbb{C}Q & \simeq 0 \qquad \forall n \geq 1 \end{cases}$$

Proof. Define the *Euler derivation* E on $\mathbb{C}Q$ by the rules that

$$E(e_i) = 0\ \forall\ 1 \leq i \leq k \qquad \text{and} \qquad E(a) = a\ \forall a \in A$$

By induction on the length $l(p)$ of an oriented path p in the quiver Q one easily verifies that $E(p) = l(p)p$. By induction one can also proof that $\mathcal{L}_E(p_0dp_1\ldots dp_n) = (l(p_0)+\cdots+l(p_n))p_0dp_1\ldots dp_n$. This implies that $\mathcal{L}_E$ is a bijection on each $\Omega^i_{\mathbb{C}V}\ \mathbb{C}Q$, where $i > 1$ and on $\Omega^0_V\ \mathbb{C}Q$, $\mathcal{L}_E$ has $\mathbb{C}V$ as its kernel. By applying the Cartan homotopy formula for $\mathcal{L}_E$, we obtain that the complex is acyclic. □

The complex $\Omega^\bullet_{\mathbb{C}V}\ \mathbb{C}Q$ induces the *relative Karoubi complex*

$$\mathsf{dR}^0_{\mathbb{C}V}\ \mathbb{C}Q \xrightarrow{d} \mathsf{dR}^1_{\mathbb{C}V}\ \mathbb{C}Q \xrightarrow{d} \mathsf{dR}^2_{\mathbb{C}V}\ \mathbb{C}Q \xrightarrow{d} \cdots$$

with

$$\mathsf{dR}^n_{\mathbb{C}V}\ \mathbb{C}Q = \frac{\Omega^n_{\mathbb{C}V}\ \mathbb{C}Q}{\sum_{i=0}^n [\ \Omega^i_{\mathbb{C}V}\ \mathbb{C}Q, \Omega^{n-i}_{\mathbb{C}V}\ \mathbb{C}Q\ ]}$$

In this expression the brackets denote supercommutators with respect to the grading on $\Omega^\bullet_V\ \mathbb{C}Q$. In the commutative case, dR^0 are the functions on the manifold and dR^1 the 1-forms.

Lemma 4.9. A $\mathbb{C}$-basis for the noncommutative functions

$$\mathsf{dR}^0_{\mathbb{C}V}\ \mathbb{C}Q \simeq \frac{\mathbb{C}Q}{[\ \mathbb{C}Q, \mathbb{C}Q\ ]}$$

are the necklace words in the quiver Q.

Proof. Let $\mathfrak{Lie}Q$ be the $\mathbb{C}$-space spanned by all necklace words w in Q and define a linear map

$$\mathbb{C}Q \xrightarrow{n} \mathfrak{Lie}Q \qquad \begin{cases} p \mapsto w_p & \text{if } p \text{ is a cycle} \\ p \mapsto 0 & \text{if } p \text{ is not} \end{cases}$$

for all oriented paths p in the quiver Q, where w_p is the necklace word in Q determined by the oriented cycle p. Because $w_{p_1p_2} = w_{p_2p_1}$ it follows that the commutator subspace $[\mathbb{C}Q, \mathbb{C}Q]$ belongs to the kernel of this map. Conversely, let

$$x = x_0 + x_1 + \ldots + x_m$$

be in the kernel where x_0 is a linear combination of non-cyclic paths and x_i for $1 \leq i \leq m$ is a linear combination of cyclic paths mapping to the same necklace word w_i, then $n(x_i) = 0$ for all $i \geq 0$. Clearly, $x_0 \in [\mathbb{C}Q, \mathbb{C}Q]$ as we can write every noncyclic path $p = a.p' = a.p' - p'.a$ as a commutator. If $x_i = a_1 p_1 + a_2 p_2 + \ldots + a_l p_l$ with $n(p_i) = w_i$, then $p_1 = q.q'$ and $p_2 = q'.q$ for some paths q, q' whence $p_1 - p_2$ is a commutator. But then, $x_i = a_1(p_1 - p_2) + (a_2 - a_1)p_2 + \ldots + a_l p_l$ is a sum of a commutator and a linear combination of strictly fewer elements. By induction, this shows that $x_i \in [\mathbb{C}Q, \mathbb{C}Q]$.

$\square$

Lemma 4.10. $\mathsf{dR}^1_{\mathbb{C}V}$ is isomorphic as $\mathbb{C}$-space to

$$\bigoplus_{(j) \xleftarrow{a} (i)} v_i.\mathbb{C}Q.v_j \; da = \bigoplus_{(j) \xleftarrow{a} (i)} (i) \dashleftarrow (j) \; d \; (j) \xleftarrow{a} (i)$$

Proof. If $p.q$ is not a cycle, then $pdq = [p, dq]$ and so vanishes in $\mathsf{dR}^1_{\mathbb{C}V}$ so we only have to consider terms pdq with $p.q$ an oriented cycle in Q. For any three paths p, q and r in Q we have the equality

$$[p.qdr] = pqdr - qd(rp) + qrdp$$

whence in $\mathsf{dR}^1_{\mathbb{C}V} \; \mathbb{C}Q$ we have relations allowing to reduce the length of the differential part

$$qd(rp) = pqdr + qrdp$$

so $\mathsf{dR}^1_{\mathbb{C}V} \; \mathbb{C}Q$ is spanned by terms of the form pda with $a \in Q_a$ and $p.a$ an oriented cycle in Q. Therefore, we have a surjection

$$\Omega^1_{\mathbb{C}V} \; \mathbb{C}Q \to \bigoplus_{(j) \xleftarrow{a} (i)} e_i.\mathbb{C}Q.e_j \; da$$

By construction, it is clear that $[\Omega^0_{\mathbb{C}V} \; \mathbb{C}Q, \Omega^1_{\mathbb{C}V} \; \mathbb{C}Q]$ lies in the kernel of this map and using an argument as in the lemma above one shows also the converse inclusion. $\square$

Using the above descriptions of $\mathsf{dR}^i_{\mathbb{C}V} \; \mathbb{C}Q$ for $i = 0, 1$ and the differential $d : \mathsf{dR}^0_{\mathbb{C}V} \; \mathbb{C}Q \to \mathsf{dR}^1_{\mathbb{C}V} \; \mathbb{C}Q$ we can define *partial differential operators* associated to any arrow $(j) \xleftarrow{a} (i)$ in Q.

$$\frac{\partial}{\partial a} \; : \; \mathsf{dR}^0_V \; \mathbb{C}Q \to e_i \mathbb{C}Q e_j \qquad \text{by} \qquad df = \sum_{a \in Q_a} \frac{\partial f}{\partial a} da$$

To take the partial derivative of a necklace word w with respect to an arrow a, we run through w and each time we encounter a we open the necklace by removing that occurrence of a and then take the sum of all the paths obtained.

Defining the *relative deRham cohomology* $\mathsf{H}^{\mathsf{n}}_{\mathsf{dR}}\ \mathbb{C}Q$ to be the cohomology of the Karoubi complex and observing that the operators $\mathcal{L}_\theta$ and i_θ on $\Omega^\bullet_{\mathbb{C}V}\ \mathbb{C}Q$ induce operators on the Karoubi complex, we have the *acyclicity result*

Theorem 4.11. The relative Karoubi complex has the following cohomology

$$\begin{cases} \mathsf{H}^{0}_{\mathsf{dR}}\ \mathbb{C}Q & \simeq\ \mathbb{C}V \\ \mathsf{H}^{\mathsf{n}}_{\mathsf{dR}}\ \mathbb{C}Q & \simeq\ 0 \qquad \forall n \geq 1 \end{cases}$$

Proof. Define $K = \oplus_{m,n}[\Omega^n_{\mathbb{C}V}\ \mathbb{C}Q, \Omega^m_{\mathbb{C}V}\ \mathbb{C}Q]$ then one verifies for the Euler derivation that

$$\mathcal{L}_E(K) \subset K \quad i_E(K) \subset K \quad \mathcal{L}_E = i_E \circ d + d \circ i_E$$

The length of a path induces a graded algebra structure on $\Omega_{\mathbb{C}V}\ \mathbb{C}Q$ and clearly K and $d^{-1}K$ are spanned by homogeneous elements. The differential of a homogeneous element is either zero or an element of the same length. Writing $x = \sum_i x_i \in d^{-1}K$ in homogeneous components we have $dx = \sum_i dx_i$ is a homogeneous decomposition. Hence, all $dx_i \in K$ whence $x_i \in d^{-1}K$. Assume that ω is a homogeneous element of length $l > 1$ in $d^{-1}K$, then

$$\begin{aligned} \omega + K =& \frac{1}{l}\mathcal{L}_E(\omega) + K \\ &= \frac{1}{l}(i_E(d\omega) + d(i_E(\omega))) + K \\ &= d(i_E(\omega)) + K \end{aligned}$$

From these facts the result follows by mimicking the proof for the cohomology of the relative differential form complex above. □

If Q is a quiver with double quiver Q^d, then we can define a canonical *symplectic structure* on the path algebra of the double $\mathbb{C}Q^d$ determined by the element

$$\omega = \sum_{a \in Q_a} da^* da \in \mathsf{dR}^2_V\ \mathbb{C}Q^d$$

As in the commutative case, ω defines a bijection between the noncommutative 1-forms $\mathsf{dR}^1_{\mathbb{C}V}\ \mathbb{C}Q^d$ and the *noncommutative vectorfields* which are defined to be the $\mathbb{C}V$-derivations of $\mathbb{C}Q^d$. This correspondence is

$$\tau : \mathsf{Der}_{\mathbb{C}V}\ \mathbb{C}Q^d \to \mathsf{dR}^1_{\mathbb{C}V}\ \mathbb{C}Q^d \qquad \text{given by} \qquad \tau(\theta) = i_\theta(\omega)$$

In analogy with the commutative case we define a derivation $\theta \in \mathsf{Der}_{\mathbb{C}V}\ \mathbb{C}Q^d$ to be *symplectic* if and only if $\mathcal{L}_\theta \omega = 0 \in \mathsf{dR}^2_{\mathbb{C}V}\ \mathbb{C}Q^d$ and denote the subspace of symplectic derivations by $\mathsf{Der}_\omega\ \mathbb{C}Q^d$. It follows from the homotopy formula and the fact that ω is a closed form, that $\theta \in \mathsf{Der}_\omega\ \mathbb{C}Q^d$ implies $L_\theta\omega = di_\theta\omega = d\tau(\theta) = 0$. That is, $\tau(\theta)$ is a closed form which by the acyclicity of the Karoubi complex shows that it must be an exact form. That is we have an isomorphism of exact sequences of $\mathbb{C}$-vectorspaces

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}V \arrow[r] \arrow[d, "="] & \mathsf{dR}^0_{\mathbb{C}V}\ \mathbb{C}Q^d \arrow[r, "d"] \arrow[d, "\simeq"] & (\mathsf{dR}^1_{\mathbb{C}V}\ \mathbb{C}Q^d)_{exact} \arrow[r] \arrow[d, "\tau^{-1}"] & 0 \\
0 \arrow[r] & \mathbb{C}V \arrow[r] & \frac{\mathbb{C}Q^d}{[\mathbb{C}Q^d,\mathbb{C}Q^d]} \arrow[r] & \mathsf{Der}_\omega\ \mathbb{C}Q^d \arrow[r] & 0
\end{tikzcd}$$

The symplectic structure ω defines a Poisson bracket on the noncommutative functions.

Definition 4.12. Let Q be a quiver and Q^d its double. The *Kontsevich bracket* on the necklace words in Q^d, $\mathsf{dR}^0_{\mathbb{C}V}\ \mathbb{C}Q^d$ is defined to be

$$\{w_1, w_2\}_K = \sum_{a \in Q_a} \left(\frac{\partial w_1}{\partial a}\frac{\partial w_2}{\partial a^*} - \frac{\partial w_1}{\partial a^*}\frac{\partial w_2}{\partial a}\right) \ \mathsf{mod}\ [\mathbb{C}Q^d, \mathbb{C}Q^d]$$

By the description of the partial differential operators it is clear that $\mathsf{dR}^0_V\ \mathbb{C}Q^d$ with this bracket is isomorphic to the necklace Lie algebra $\mathfrak{Lie}Q$.

The symplectic derivations $\mathsf{Der}_\omega\ \mathbb{C}Q^d$ have a natural Lie algebra structure by commutators of derivations. We will show that $\tau^{-1} \circ d$ is a Lie algebra morphism.

For every necklace word w we have a symplectic derivation $\theta_w = \tau^{-1}dw$ defined by

$$\begin{cases} \theta_w(a) & = -\frac{\partial w}{\partial a^*} \\ \theta_w(a^*) & = \frac{\partial w}{\partial a} \end{cases}$$

With this notation we get the following interpretations of the Kontsevich bracket

$$\{w_1, w_2\}_K = i_{\theta_{w_1}}(i_{\theta_{w_2}}\omega) = \mathcal{L}_{\theta_{w_1}}(w_2) = -\mathcal{L}_{\theta_{w_2}}(w_1)$$

where the next to last equality follows because $i_{\theta_{w_2}}\omega = dw_2$ and the fact that $i_{\theta_{w_1}}(dw) = \mathcal{L}_{\theta_{w_1}}(w)$ for any w. More generally, for any V-derivation θ and any necklace word w we have the equation

$$i_\theta(i_{\theta_w}\omega) = \mathcal{L}_\theta(w).$$

When we look at the image of the Kontsevich bracket under $\tau^{-1}d$, we obtain the following

$$\begin{aligned}
\tau^{-1}d\{w_1,w_2\}_K &= \tau^{-1}d\mathcal{L}_{\theta_{w_1}}w_2\\
&= \tau^{-1}\mathcal{L}_{\theta_{w_1}}dw_2\\
&= \tau^{-1}\mathcal{L}_{\theta_{w_1}}i_{\theta_{w_2}}\omega\\
&= \tau^{-1}([\mathcal{L}_{\theta_{w_1}},i_{\theta_{w_2}}]+i_{\theta_{w_2}}\mathcal{L}_{\theta_{w_1}})\omega\\
&= \tau^{-1}i_{[\theta_{w_1},\theta_{w_2}]}\omega\\
&= [\theta_{w_1},\theta_{w_2}]
\end{aligned}$$

Above we made use of the fact that $\mathcal{L}_\theta$ commutes with d, and the defining equation $dw_2 = i_{\theta_{w_2}}\omega$. In the fourth line we omitted the last term because θ_{w_1} is a symplectic derivation. Finally lemma 4.6 enabled us to transform the commutator in i and $\mathcal{L}$ to of commutator of the derivations θ_{w_1} and θ_{w_2}. This calculation concluded the proof of :

Theorem 4.13. With notations as before, $\mathsf{dR}^0_V\ \mathbb{C}Q^d$ with the Kontsevich bracket is isomorphic to the necklace Lie algebra $\mathfrak{Lie}Q$, and the sequence

$$0 \longrightarrow \mathfrak{Lie}_Q \xrightarrow{\tau^{-1}d} \mathsf{Der}_\omega\ \mathbb{C}Q^d \longrightarrow 0$$

is an exact sequence (hence a central extension) of Lie algebras.

4.4 The moment map and the preprojective algebra

In this section we study the moment map for the GL_α-action on the representation space $\mathsf{Rep}_\alpha Q^d$.

First of all the natural map from $\mathfrak{gl}_\alpha := \bigoplus \mathsf{Mat}_{\alpha_i\times\alpha_i}(\mathbb{C})$ to $\mathsf{VectRep}_\alpha Q^d$ identifies with every $g := (g_{v_1},\ldots,g_{v_k})$ the following vectorfield

$$X_g : \mathsf{Rep}_\alpha Q^d \to \mathsf{Rep}_\alpha Q^d : W \mapsto W' \text{ with } W'_a = g_{t(a)}W_a - W_a g_{s(a)}$$

Using the symplectic structure, this vectorfield can be identified with the 1-form df_g coming from the function

$$f_g : \mathsf{Rep}_\alpha Q^d \to \mathbb{C} : W \mapsto \sum_{a\in A}\mathsf{Tr}(g_{s(a)}W_aW_{a^*}) - \mathsf{Tr}(W_a^*W_ag_{t(a)}).$$

Using trace pairing we can identify $\mathfrak{gl}_\alpha^*$ with $\mathfrak{gl}_\alpha$ and hence we come to the following formula of the moment map:

$$\mu : \mathsf{Rep}_\alpha Q^d \to \mathfrak{gl}_\alpha : W \mapsto \mu(W) \text{with } \mu(W)_v = \sum_{a\in A,t(a)=v}W_aW_{a^*} - \sum_{a\in A,s(a)=v}W_{a^*}W_a.$$

Notice that the image of μ is not the whole space $\mathfrak{gl}_\alpha$ but the subspace $\mathfrak{pgl}_\alpha := \{g \in \mathfrak{gl}_\alpha | \sum_v \mathsf{Tr} g_v = 0\}$ because the scalar matrices act trivially on $\mathsf{Rep}_\alpha Q^d$.

Definition 4.14. We will call $\lambda \in \mathbb{C}V$ a weight. For every weight λ we define the *deformed preprojective algebra* of the quiver Q to be

$$\Pi_\lambda Q := \frac{\mathbb{C}Q^d}{(c-\lambda)}$$

where c is the *commutator element*

$$\sum_{a \in A} [a, a^*].$$

A weight λ can also be seen as a vector $(\lambda_{v_1}, \dots, \lambda_{v_k})$ indexed by the vertices of Q.

If we look at the α-dimensional representations of $\Pi_\lambda Q$ we can see them as representations $W \in \mathsf{Rep}_\alpha Q^d$ satisfying the equation

$$\sum_{a \in A, t(a)=v} W_a W_{a^*} - \sum_{a \in A, s(a)=v} W_{a^*} W_a = \lambda_v \mathbf{1}_{\alpha_v}$$

for every vertex v. This is the same as stating that the image of the moment map $\mu(W)$ must be equal to $\hat{\lambda} := (\lambda_{v_1} \mathbf{1}_{\alpha_{v_1}}, \dots, \lambda_{v_k} \mathbf{1}_{\alpha_{v_k}})$. Therefore the representation space of the deformed preprojective algebra can be seen as a fiber of the moment map:

$$\mathsf{Rep}_\alpha \Pi_\lambda Q := \mu^{-1}(\hat{\lambda})$$

Because the image of μ is traceless and $\mathsf{Tr}\hat{\lambda} = \sum_v \mathsf{Tr}(\lambda_v \mathbf{1}_{\alpha_v}) = \sum_v \lambda_v \alpha_v$ we can conclude that there are no α-dimensional representations of $\mathsf{Rep}_\alpha \Pi_\lambda Q$ unless that $\lambda \cdot \alpha = 0$. If $\mathsf{Rep}_\alpha \Pi_\lambda Q$ is not empty it is not so that is necessary a smooth variety, in the next paragraphs we will give a sufficient condition for this to be the case.

For every weight λ we define Σ_λ to be the set of dimension vectors of Q for which there exist simple represetations of $\Pi_\lambda Q$. The description of this set is rather complicated and is given by Crawley-Boevey

Theorem 4.15. Let Δ_λ^+ be the set of positive roots of Q (i.e. the dimension vectors α for which there exist indecomposible representations in $\mathsf{Rep}_\alpha Q$) such that $\lambda \cdot \alpha = 0$. Define Σ_λ to be the set of positive roots $\alpha \in \Delta_\lambda^+$ such that for every decomposition $\alpha = \beta_1 + \dots + \beta_r,\ \beta_i \Delta_\lambda^+$

$$1 - \chi_Q(\alpha, \alpha) < 1 - \chi_Q(\beta_1, \beta_1) + \dots + 1 - \chi_Q(\beta_r, \beta_r).$$

For every $\alpha \in \Sigma_\lambda$, $\mathsf{Rep}_\alpha \Pi_\lambda$ contains simple representations and both $\mathsf{Rep}_\alpha \Pi_\lambda$ and $\mathsf{iss}_\alpha \Pi_\lambda$

On this set there is a natural ordering $\beta \leq \alpha \Leftrightarrow \forall v \in V : \beta_v \leq \alpha_v$. Let now Σ_λ^{min} be the set of minimal dimension vectors in Σ_λ for this ordering.

Theorem 4.16. If $\alpha \in \Sigma_\lambda^{min}$ then $\mathsf{Rep}_\alpha\Pi_\lambda$ and $\mathsf{iss}_\alpha\Pi_\lambda$ are both smooth varieties and the quotient map

$$\mathsf{Rep}_\alpha\Pi_\lambda \twoheadrightarrow \mathsf{iss}_\alpha\Pi_\lambda$$

makes $\mathsf{Rep}_\alpha\Pi_\lambda$ into a principal PGL_α-bundle.

Proof. Suppose $W \in \mathsf{Rep}_\alpha\Pi_\lambda$ then W must be a simple representation because the Jordan-Hölder components of W can only be of dimension α and hence W has only one component. Therefore the orbit $\mathcal{O}_W$ is closed. If we compute the differential of the complex moment map in W we see that

$$d\mu_W(V) : \mathfrak{gl}_\alpha \to \mathbb{C} : g \mapsto df_{gW}(V) = \omega(gW, V)$$

This means that the image of $d\mu_W$ can be seen as all elements from $(\mathfrak{gl}_\alpha gW)^*$. This space has dimension $Dim\mathfrak{gl}_\alpha - Dim\mathsf{Stab}W$ which dus not depend on W because we know that for simple representations $\mathsf{Stab}W = \mathbb{C}^*$. Al this implies that $\mu^{-1}(\lambda)$ is a smooth variety.

For every W the stabilizer subgroup in GL_α is the group of the scalar matrices so $\mathcal{O}_W \cong \mathsf{PGL}_\alpha Q$. Therefore the projection map

$$\mathsf{Rep}_\alpha\Pi_\lambda \twoheadrightarrow \mathsf{iss}_\alpha\Pi_\lambda$$

has isomorphic fibers and is a principal fibration. Because the total space $\mathsf{Rep}_\alpha\Pi_\lambda$ is smooth the base space $\mathsf{iss}_\alpha\Pi_\lambda$ must also be smooth. $\square$

So if α is a minimal element of Σ_λ, $\mathsf{iss}_\alpha\Pi_\lambda$ is a smooth variety. This is however not the whole story, we can prove that $\mathsf{iss}_\alpha\Pi_\lambda$ is a coadjoint orbit of an infinite dimensional Lie algebra.

As we have already seen before the action of GL_α on $\mathsf{Rep}_\alpha Q^d$ is Hamiltonian and we have the following diagram

$$\begin{tikzcd} & \mathfrak{gl}_\alpha \arrow[dl, dotted, "\mu^*"'] \arrow[d] \\ \mathbb{C}[\mathsf{Rep}_\alpha Q^d] \arrow[r] & \mathsf{Vect}_\omega(\mathsf{Rep}_\alpha Q^d). \end{tikzcd}$$

The dual space of the Lie algebra $\mathfrak{Poiss}_\alpha Q$, can also be considered as a Poisson manifold equipped with the Kirillov-Konstant bracket on $\mathbb{C}[\mathfrak{Poiss}_\alpha Q^*]$ coming forth from the Lie bracket on $\mathfrak{Poiss}_\alpha Q \subset \mathbb{C}[\mathfrak{Poiss}_\alpha Q^*]$.

Evaluation at a point of $\mathsf{iss}_\alpha Q^d$ defines a linear function on $\mathfrak{Poiss}_\alpha Q$ and therefore we can embed $\mathsf{iss}_\alpha Q^d$ as a Poisson manifold in $(\mathfrak{Poiss}_\alpha Q)^*$. Combining this with the injection of $\mathsf{iss}_\alpha \Pi_\lambda$ in Q. we have the diagram

$$\mathsf{iss}_\alpha \Pi_\lambda Q \hookrightarrow \mathsf{iss}_\alpha Q^d \hookrightarrow (\mathfrak{Poiss}_\alpha Q)^*.$$

This is not only an embedding as varieties but also as Poisson manifolds because the bracket structure on $\mathbb{C}[\mathsf{iss}_\alpha Q^d]$ induces the bracket on the quotient $\mathbb{C}[\mathsf{iss}_\alpha \Pi_\lambda Q]$.

Dualizing the ordinary adjoint Lie action of $\mathfrak{Poiss}_\alpha Q$ on itselfs gives us a coadjoint action of $\mathfrak{Poiss}_\alpha Q$ on $(\mathfrak{Poiss}_\alpha Q)^*$. The following theorem connects this action to the representation space of the deformed preprojective algebras.

Theorem 4.17. If $\alpha \in \Sigma_\lambda^{min}$ then $\mathsf{iss}_\alpha \Pi_\lambda Q \hookrightarrow (\mathfrak{Poiss}_\alpha Q)^*$ is a closed coadjoint orbit of the infinite dimensional Lie algebra $\mathfrak{Poiss}_\alpha Q$.

Proof. $\mathsf{iss}_\alpha \Pi_\lambda$ is a smooth symplectic variety so the symplectic vectorfields span in every point p the total tangent space $\mathrm{T}_p \mathsf{iss}_\alpha \Pi_\lambda$. This implies that the Lie algebra of symplectic vector fields works infinitesimally transitive on $\mathsf{iss}_\alpha \Pi_\lambda$, and the same holds thus for $\mathbb{C}[\mathsf{iss}_\alpha \Pi_\lambda], \{,\} = \mathfrak{Poiss}_\alpha Q$.

Using the evaluation map we can embed $\mathsf{iss}_\alpha \Pi_\lambda$ in $\mathbb{C}[\mathsf{iss}_\alpha \Pi_\lambda]^*$, the space $T_p \mathsf{iss}_\alpha \Pi_\lambda$, seen as the p-derivations, is also embedded in $\mathbb{C}[\mathsf{iss}_\alpha \Pi_\lambda]^*$. Now if we look at the coadjoint action of $h \in \mathbb{C}[\mathsf{iss}_\alpha \Pi_\lambda]$ on p viewed as an element of $\mathbb{C}[\mathsf{iss}_\alpha \Pi_\lambda]^*$ we see that

$$hp : \mathbb{C}[\mathsf{iss}_\alpha \Pi_\lambda] \to \mathbb{C} : f \to \{h, f\}(p) = [\tau^{-1} dh] f(p)$$

can be identified with the tangent vector coming from the symplectic vectorfield $\tau^{-1} dh$. Therefore $\mathsf{iss}_\alpha \Pi_\lambda$ can be viewed as an invariant subspace of $\mathbb{C}[\mathsf{iss}_\alpha \Pi_\lambda]^*$ for the coadjoint action. This subspace contains only one orbit because it is irreducible, by theorem 4.16 $\square$

The Lie algebra $\mathfrak{Poiss}_\alpha Q$ depends on the dimension vector α. It would be interesting to see all the $\mathsf{iss}_\alpha \Pi_\lambda Q$ for all α as coadjoint orbits of the same lie algebra. To do this we need a bigger Lie algebra. An interesting candidate for this job is the necklace Lie algebra:

$$\mathfrak{Lie} Q = \frac{\mathbb{C} Q^d}{[\mathbb{C} Q^d, \mathbb{C} Q^d]}$$

has for every dimension vector a trace map

$$\mathsf{Tr}_\alpha : \mathfrak{Lie} Q \to \mathfrak{Poiss}_\alpha Q$$

which is a Lie homomorphism. Because the image of Tr_α generates $\mathfrak{Poiss}_\alpha Q$ as a Poisson algebra, the elements of $\mathsf{Tr}_\alpha \mathfrak{Lie} Q$ are enough to separate points of $\mathsf{iss}_\alpha Q^d$. Therefore the map Tr_α^* is an injection and we have the diagram

$$\mathsf{iss}_\alpha \Pi_\lambda Q \hookrightarrow \mathsf{iss}_\alpha Q^d \xhookrightarrow{\mathsf{Tr}_\alpha^*} (\mathfrak{Lie} Q)^*.$$

and we can conclude with the theorem

Theorem 4.18. If $\alpha \in \Sigma_\lambda^{min}$ then the embeddings

$$\mathsf{iss}_\alpha \Pi_\lambda Q \hookrightarrow \mathsf{iss}_\alpha Q^d \hookrightarrow (\mathfrak{Poiss}_\alpha Q)^* \hookrightarrow (\mathfrak{Lie} Q)^*$$

make $\mathsf{iss}_\alpha \Pi_\lambda Q$ into a closed coadjoint orbit of the infinite dimensional Lie algebra $\mathfrak{Lie} Q$.

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