Chapter 1

Covers and Fundamental Groups

In this chapter we suppose that a manifold is always connected.

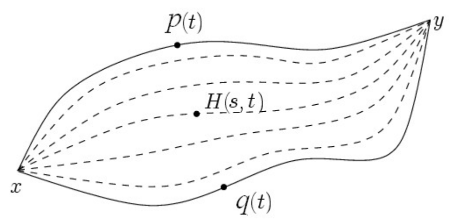
1.1 The Fundamental Group

Definition 1.1. By a path p in M we mean a continuous map $p : [0,1] \to M$. A loop is defined as a path with $p(0) = p(1)$.

Two paths p, q are said to be homotopic ($p \equiv q$) if they can be deformed into each other while keeping the end points the same. Mathematically there exists a continuous map

$$H : [0,1] \times [0,1] \to M \text{ such that } \forall t \in [0,1] : \begin{cases} p(t) = H(0,t), q(t) = H(1,t) \\ H(t,0) = p(0), H(t,1) = p(1) \end{cases}$$

The relation being homotopic to each other is an equivalence relation, so we can talk about the homotopy class of a given path.



Exercise 1.2. Prove that homotopy is indeed an equivalence relation (i.e. reflexive, transitive, symmetric).

Definition 1.3. If $p(1) = q(0)$ we can concatenate two paths to obtain a new path:

$$q * p : [0,1] \to M : t \mapsto \begin{cases} p(2t) & t \leq \frac{1}{2} \\ q(2t-1) & q \geq \frac{1}{2} \end{cases}$$

Exercise 1.4. Prove that homotopy compatible with concatenation:

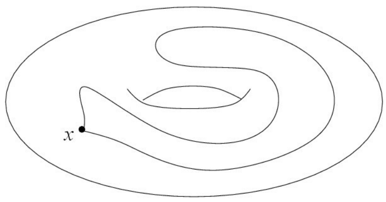
$$p_1 \equiv p_2, q_1 \equiv q_2 \Rightarrow q_1 * p_1 \equiv q_2 * p_2$$

From now one we will identify a path with its homotopy class.

This multiplication of paths is interesting because it allows us to define a group:

Definition 1.5. if x is a point in M we define the fundamental group $\pi_1(M, x)$ as all the homotopy classes of loops that start in M. We define the multiplication in the group to be the concatenation of loops.

Exercise 1.6. Check that $\pi_1(M, x)$ is indeed a group. Its neutral element is the trivial path $e_x : t \mapsto x$. The inverse of a path is $p^{-1} : t \mapsto p(1-t)$.



Although the definition of depends on the point x, fundamental groups coming from different points in the same manifold are isomorphic. If $x, y \in M$ and p is a path from x to y then

$$\pi_1(x, M) \to \pi_1(y, M) : q \mapsto p * q * p^{-1}$$

is an isomorphism.

Example 1.7. In $\mathbb{R}^n$ every loop is trivial. Suppose $p(0) = p(1) = 0$ then define the homotopy $e_0 \equiv p$ by

$$H : [0,1] \times [0,1] : (s,t) \mapsto sp(t)$$

Therefore $\pi_1(\mathbb{R}^n) = 1$.

The same holds for the n-dimensional sphere with $n > 1$: $\mathbb{S}_n \subset \mathbb{R}^{n+1} : x_0^2 + \cdots + x_n^2 = 1$. This is proved as follows: take a loop and consider a point m not on the loop. Now we can do a projection of $\mathbb{S}_n$ onto a hyperplane of $\mathbb{R}^{n+1}$ this gives us a loop in this hyperplane which is trivial. Project the homotopy of this trivial loop back to the sphere and this gives us a trivialization of the loop on the sphere.

For $n = 1$ there do exist nontrivial loops: consider the loop

$$\ell_k : t \mapsto (\cos 2\pi kt, \sin 2\pi kt).$$

These are all nontrivial if $k \neq 0$. Also we have that $\ell_k * \ell_l \equiv \ell_{k+l}$ and therefore we might guess that $\pi_1(\mathbb{S}_1) = \mathbb{Z}$. In the next section we will prove this rigorously.

With all this in mind we can now state the Poincaré conjecture:

Conjecture 1.8. *A compact n-dimensional* ($n \geq 2$) *manifold has trivial fundamental group if and only if it is diffeomorphic to the n-sphere.*

A manifold with trivial fundamental group is also called simply connected.

The conjecture was originally formulated by Henri Poincaré in the early 1900's. It was know by Poincaré to be true for $n = 2$. In The 40-60's it was proved by Smale, Stallings, Wallace and Zeeman for high dimensions $n \geq 5$. Later on in the 80's Freedman added the case $n = 4$ leaving only the three dimensional case unsolved. Finally in 2002 Grisha Perelmann proved this last part of the conjecture. In these course notes we are going to have a look at several of the key ingredients needed in this final part of the proof.

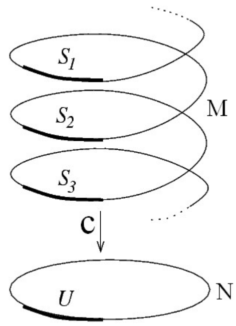
Miniature 1: Henri Poincaré (1854 - 1912)

Jules Henri Poincaré was one of France's greatest mathematicians and theoretical physicists, and a philosopher of science. He is often described as a polymath, a person who excels in multiple fields, particularly in both arts and sciences.
As a mathematician and physicist, he made many original fundamental contributions to pure and applied mathematics, mathematical physics, and celestial mechanics. He was responsible for formulating the Poincaré conjecture, one of the most famous problems in mathematics.

1.2 Covers

Definition 1.9. A map $c : M \to N$ is called a cover of N if for every $x \in M$ we can find an open neighborhood S_x such that $c|_{U_x}$ is a diffeomorphism. Such neighborhoods are called small.

Stated in another way, for every $y \in N$ we can find a neighborhood U such that $c^{-1}(U)$ is the disjoint union of (a possibly infinite number of) open sets S_i such that $c|_{S_i} : S_i \to U$ is a diffeomorphism.



An example of a cover is the map $c : \mathbb{R} \to \mathbb{S}_1 : t \mapsto (\cos t, \sin t)$ the open neighborhoods are f.i. $]x-1, x+1[$ because every point in such an interval is mapped to a different point in the circle. The map $c : \mathbb{S}_2 \to \mathbb{P}_2 : (x, y, z) \mapsto (x : y : z)$ is also a cover, the small neighborhoods can be taken $U_x := B(x, \frac{1}{2}) \cap \mathbb{S}_2$.

The connection between paths and covers is captured in the lemma below

Lemma 1.10. *If $c : M \to N$ is a cover, $p : [0,1] \to N$ a path and $x \in M$ a point such that $c(x) = p(0)$ then there is a unique lifted path $\tilde{p} : [0,1] \to M$ such that $c\tilde{p} = p$ and $\tilde{p}(0) = x$. Moreover if $p \equiv q$ then also $\tilde{p} \equiv \tilde{q}$.*

Proof. If the image of p is contained in the image under c of a small neighborhood of x then this statement is trivial because $\tilde{p}$ must then be equal to $c|_{U_x}^{-1}p$. In general this is not always the case but we can find a $t_1 > 0$ such that $p([0, t_1]) \subset U_x$, so we can lift at least a small bit of p. Now let t_m be the supremum of

$$T = \{t | p_{[0,t_m]} \text{ can be lifted } \}$$

We have to show that $t_m = 1$. First note that is T is a closed interval because if $T = [0, t_m[$ then we can define $\tilde{p}(t_m) := \lim_{t \to t_m} \tilde{p}_m$. Now if $t_m < 1$ then we can find a small neighborhood U of $\tilde{p}(t_m)$ and a $t > t_m$ such that $p([t_m, t]) \subset cU$ and we can lift this part of the path as well.

To prove that lifts of homotopic path are again homotopic we have to lift the homotopy H to M. Because $H(s, --)$ is a path for every s we can lift it uniquely to M, the lift of H must hence be the 'union' of all these lifts. We only have to prove that this $\tilde{H}$ is continuous. $\square$

Although we can lift every path uniquely it is not necessarily true that the lift of a loop is again a loop: it might be possible that $\tilde{p}(1)$ is another point in $c^{-1}(p(1))$. This implies that the fundamental group of M can be seen as the subgroup of the fundamental group of N consisting of the homotopy classes whose lift is again a loop.

If M is a simply connected manifold then the elements of the fundamental group are in one to one correspondence to the elements of $c^{-1}(x)$ with $x \in N$. Fix an $m \in c^{-1}(x)$ then for every $m' \in c^{-1}(x)$ there exist a unique homotopy class of paths from m to m'. This homotopy class projects to a homotopy class of loops in $\pi_1(N)$. Vice versa every loop in $\pi_1(N)$ lifts to a path from m to some point in $c^{-1}(x)$ which is uniquely determined by its end point because M is simply connected.

This observation will enable us to obtain another description of the fundamental group.

Definition 1.11. Let $c : M \to N$ be a cover. A deck transformation is a diffeomorphism $d : M \to M$ such that $cd = c$. We denote the set of deck transformations by $\mathcal{D}(M/N)$. This set has the structure of a group under the standard composition of maps.

The main theorem of covers now holds

Theorem 1.12. *If $c : M \to N$ is a cover and M is simply connected then*

$$\pi_1(N) \cong \mathcal{D}(M/N)$$

Proof. Fix a point $m \in M$. Using this point we will construct maps between the two groups in both directions and show that these are each other's inverse.

To go from a deck transformation d to a loop in N we construct a path p from m to $d(m)$. This path is unique up to homotopy and hence its image under c determines a unique element $\ell_d := cp \in \pi_1(N, c(m))$.

If we have a loop ℓ in N starting from $c(m)$ we can lift this loop to a path $\tilde{l}$ starting in m. We define a deck transformation in the following way. If $x \in M$, let p be a path from x to m. The path cp^{-1} runs in N from $c(m)$ to $c(x)$. We can lift this path uniquely to a path $\tilde{p}^{-1}$ starting in $\tilde{\ell}(1)$. The end point of this lift i.e. $\tilde{x} := \tilde{p}^{-1}(1)$ only depends on the homotopy class of ℓ (and not of p because M is simply connected). The assignment $d_\ell : x \to \tilde{x}$ is a deck transformation because $\tilde{x}$ sits inside $c^{-1}(c(x))$.

We now have to prove that $\ell_{d_\ell} = \ell$ and $d_{\ell_d} = d$ The first one is obvious because the homotopy class of ℓ_{d_ℓ} is uniquely determined by $\tilde{\ell}_{d_\ell}(1)$ which is by definition

$d_\ell(m) = \tilde{\ell}(1)$. The second one holds because $d_{\ell_d}(x)$ is the endpoint of the lift of the path $cp^{-1}\ell_d$ such that it starts in m, or the endpoint of the lift of cp^{-1} such that it starts in $d(m)$. But this is also true for $d(x)$ because the path from $d(m)$ to $d(x)$ (which is unique up to homotopy) must be homotopic to the lift of of cp^{-1}. □

The theorem above only works if we have a simply connected cover of N. However we can also prove that every manifold has such a simply connected cover. Such a cover is called a universal cover.

Theorem 1.13. *every manifold has a universal cover.*

Proof. Let N be a manifold and chose a point $n \in N$. We define $\tilde{N}$ as the set of all homotopy classes of paths in N starting from n. The projection $c : \tilde{N} \to N$ maps every path to its end point. To give $\tilde{N}$ a differential structure we first cover N wit charts that are simply connected. Given a path $p \in \tilde{N}$ and a chart $\phi : U \to \mathbb{R}^n$ on N that contains $p(1)$ we construct a chart $\tilde{\phi}$ the source $\tilde{U}$ is the set of path qp where q is a path in U and $\tilde{\phi}(qp) : \phi(1)$. This definition is well defined as q is uniquely defined by its endpoint because U is simply connected. Note that the $\tilde{U}$ are the small neighborhoods for the cover map $\tilde{N} \to N$.

Now we have to prove that $\tilde{N}$ is simply connected. If this were not the case than there is a nontrivial loop $\tilde{\ell}$ in $\tilde{N}$. The image under the cover map gives us a loop in N. This loop is trivial because otherwise the end point of its lift would not be $\tilde{\ell}(1)$. Lifts preserve homotopy so $\tilde{\ell}$ must also be trivial. □

Example 1.14. With this knowledge we can easily determine the fundamental group of the circle and the torus. The universal cover of the circle is the map $\mathbb{R} \to \mathbb{S}_1 : (\cos t, \sin t)$. The deck transformations are the maps $\mathbb{R} \to \mathbb{R} : x \mapsto x + 2k\pi$ so $\pi_1(\mathbb{S}_1) = \mathbb{Z}$. In a similar way the fundamental group of a torus is $\mathbb{Z}^2$. The universal cover of $\mathbb{P}_2$ is $\mathbb{S}_2 \to \mathbb{P}_2$ so the fundamental group must be $\mathbb{Z}_2$ (the only nontrivial deck transformation is the inversion $\vec{x} \to -\vec{x}$.

Chapter 2

Vectors, Tensors and Metrics

2.1 Tensors and Bundles

Definition 2.1. A trivial vector bundle over a base manifold M is a manifold of the form $M \times \mathbb{R}^k$ this object has a natural projection map π onto M and every fiber has the structure of an k-dimensional vector space.

A vector bundle over a manifold M consists of a smooth map $\pi : E \to M$ such that every fiber $\pi^{-1}(x), x \in M$ has the structure of a k-dimensional vector space. It must also have a cover $\{U_i \subset M\}$ and a set of identifications $\phi_i : \pi^{-1}(U_i) \to U \times \mathbb{R}^k$ is called a local trivialisations. The set of local trivializations must also satisfy a compatibility condition: if $x \in U_i \cap U_j$ then $\Phi_i^{-1}\Phi_j$ must be a linear isomorphism of the vector space $\pi^{-1}(x)$.

A (global) section of a vector bundle is a smooth map $s : M \to E$ such that $\forall x \in M : \pi s(x) = x$. A section assigns to each point in the base a vector in its fiber. If s is only defined on an open set $U \subset M$ then we call s a local section. The global sections of E form a module over the ring of smooth functions from M to $\mathbb{R}$: $C^\infty(M)$. This module will be denoted by $\Gamma(E)$.

Instead of giving local trivializations one can also cover M with open parts U for which there exists k local sections $s_1, \dots s_k$ such that in every point $p \in U$, $s_1(p), \dots, s_k(p)$ form a basis for the fiber $\pi^{-1}(p)$. The corresponding local trivialization is the expression of every vector $v \in \pi^{-1}(p)$ into its coordinates for the basis $s_1(p), \dots, s_k(p)$.

Example 2.2. The first nontrivial example of a vector bundle is the Moebius strip. Take M to be the circle $i.e.\mathbb{P}_1$ and define

$$V := \{((x : y), \lambda x, \lambda y) \in \mathbb{P}_1 \times \mathbb{R}^2\}$$

For every point in $\mathbb{P}_1$ the fiber is isomorphic to the corresponding line in $\mathbb{R}^2$ so each fiber is a one dimensional vector space. Although $V \neq \mathbb{P}_1 \times \mathbb{R}$ we find two open subsets of V that are trivial:

$$V = V_x \cup V_y \text{ with } V_x = \{(x:1), \lambda x, \lambda)\} \cong \{(x, \lambda)\} = \mathbb{R} \times \mathbb{R}$$

Definition 2.3. Given two smooth curves $\gamma_1, \gamma_2 : \mathbb{R} \to M$ we say that they are tangent in 0 if $\gamma_1(0) = \gamma_2(0)$. there is a chart $\phi : U \to \mathbb{R}^n$ such that

$$\frac{d\phi\gamma_1(t)}{dt}|_{t=0} = \frac{d\phi\gamma_2(t)}{dt}|_{t=0}.$$

In this way we define the tangent vector v_γ as an equivalence class of all curves that are tangent to γ in zero. Being sloppy we also use the notation $\gamma'(0)$ or $\frac{d\gamma}{dt}(0)$ instead of v_γ.

The set of all tangent vectors is denoted by TM. It has a natural projection map: $\pi : TM \to M : v_\gamma \mapsto \gamma(0)$. Using a chart ϕ we can identify each tangent vector with a unique point couple

$$(\gamma(0), \frac{d\phi\gamma(t)}{dt}|_{t=0}) \in U \times \mathbb{R}^n.$$

These identifications give us the local trivializations that turn TM into a vector bundle. The fiber of a point p is denoted by T_pM. A section of the tangent bundle is call a vector field.

Every chart gives us n (local) vector fields with local trivializations

$$\partial_i(p) \equiv (p, (0, \dots, 1, \dots, 0))$$

In every point $p \in U$ these n vector fields give a basis T_pM.

A vector field V can be applied to a function $f : M \to \mathbb{R}$ to obtain a new function

$$V(f) : M \to \mathbb{R} : p \mapsto \frac{df(\gamma(t))}{dt}|_{t=0} \text{ if } V(p) = v_\gamma$$

In this way we can see a vector field as a derivation on the ring of smooth functions $C^\infty(M)$. Every such derivation $X : C^\infty(M) \to C^\infty(M)$ can be seen as a vector field: if $\phi : U \to \mathbb{R}^n$ is a chart then $X = X(\phi_i)\partial_i$.

The identification between vector fields and derivations allows us to define an operation between vector fields: the commutator

$$[X, Y] = X \circ Y - Y \circ X : C^\infty(M) \to C^\infty(M).$$

It is easy to check that this is again a derivation and hence a vector field. Note that if ϕ is a chart, the commutators of the local vector fields ∂_i is zero.

$$[\partial_i, \partial_j] = 0$$

Vector fields also have a nice interpretation in terms of ordinary first order differential equations. Given a vector field V, we can try to define curves γ on M such that for each t in an interval I

$$\gamma'(t) = V(\gamma(t))$$

As a consequence of the unique existence of solutions for linear differential equations, we can find for every x a unique curve $\gamma_x :]-\epsilon, +\epsilon[\to M$ such that

$$\gamma_x(0) = x \text{ and } \gamma_x'(t) = V(\gamma_x(t)) \qquad (t \in]-\epsilon, +\epsilon[)$$

These curves are called flow curves of the vector field V and partition M into equivalence classes. It is not always possible to extend the interval $]-\epsilon, +\epsilon[$ to the whole of $\mathbb{R}$. If one can do this one obtains a one-parameter family of automorphisms of M:

$$\exp(tV) : M \to M : x \mapsto \gamma_x(t).$$

Vice versa every one parameter family of automorphisms $\phi_t : M \to M$ such that $\phi_0 = 1$ gives us a vectorfield

$$X : M \to TM : p \to \frac{d}{dt}\phi_t(p).$$

Starting from the tangent bundle we can define new bundles:

1. the cotangent bundle is the union of all dual spaces $(T_pM)^*$. For a given chart ϕ and point p, we define $dx^1(p), \dots, dx^n(p)$ to be the dual basis of $\partial_1(p), \dots, \partial_i(p)$. Coordinatization in terms of these bases provides us with a local trivialization. Sections of this bundle are called 1-forms and the space of 1-forms is denoted by $\Omega_1(M)$.

   Given a 1-form ω and a vector field V we can *contract* them to get a function $\omega(V) : M \to \mathbb{R} : p \mapsto \omega(p)(V(p))$. In this way we can see a one-form as a $\mathbb{C}^\infty$-linear map from $\mathsf{Vect}(M)$ to $\mathbb{C}^\infty$. Conversely every such $\mathbb{C}^\infty$-linear map can be seen as a one-form:

$$\Omega_1(M) := \mathsf{Hom}_{C^\infty(M)}(\mathsf{Vect}(M), C^\infty(M)).$$

2. The (k,l)-tensor bundle has as fibers the vector spaces $(T_pM)^{\otimes k} \otimes (T_pM)^{*\otimes l}$. For a given chart the we can define local sections $\partial_{i_1} \otimes \cdots \otimes \partial_{i_k} \otimes dX^{j_1} \otimes \cdots \otimes dX^{j_l}$. The section of this bundle are called tensor field or more sloppy tensors. The space of tensor fields is denoted by $\mathsf{Ten}^{k,l}(M)$ and is a $C^\infty(M)$ -module in the standard way. Every tensor can be be expressed in local coordinates using the index notation

$$T := \sum_{i_1,\ldots,j_l} T^{i_1\ldots i_k}_{j_1\ldots j_l} \partial_{i_1} \otimes \cdots \otimes \partial_{i_k} \otimes dX^{j_1} \otimes \cdots \otimes dX^{j_l}$$

It is costume to write only the indexed coefficients.

The product of a k_1, l_1-tensor and a k_2, l_2-tensor can be seen as a k_1+k_2, l_1+l_2 tensor in the natural way

$$(T \otimes U)^{i_1\ldots i_{k_1+k_2}}_{j_1\ldots j_{l_1+l_2}} := T^{i_1\ldots i_{k_1}}_{j_1\ldots j_{l_1}} U^{i_{k_1+1}\ldots i_{k_1+k_2}}_{j_{l_1+1}\ldots j_{l_1+l_2}}$$

Note that this product is not commutative. Another operation that one can do with tensors is contraction. This transforms a k,l-tensor in a $k-1, l-1$-tensor. The trick is to define

$$(C_{ab}T)^{i_1\ldots i_{k-1}}_{j_1\ldots j_{l-1}} := \sum_\mu T^{i_1\ldots i_{a-1}\mu i_a \ldots i_{k-1}}_{j_1\ldots j_{b-1}\mu j_b j_{l-1}}.$$

Usually the summation sign is omitted and summation is assumed for every index that appears twice. This is called the Einstein convention.

Keep in mind that the coefficients depend on the choice of a chart. The transition between two charts can be nicely expressed using the summation convention.

$$T^{i_1\ldots i_k}_{j_1\ldots j_l} = (T')^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l} \frac{\partial Y^{i_1}}{\partial X^{\mu_1}} \cdots \frac{\partial Y^{i_k}}{\partial X^{\mu_k}} \frac{\partial X^{\nu_1}}{\partial Y^{j_1}} \cdots \frac{\partial X^{\nu_k}}{\partial Y^{j_k}}$$

The Y^i and X^i are the coordinate functions of the two charts.

As with 1-forms one can see tensors as C^∞-linear maps: a (k,l)-tensor T is a multilinear map

$$\begin{aligned} T &: \Omega_1(M)^k \times \mathsf{Vect}(M)^l \to C^\infty \\ &: (\omega_1, \ldots, \omega_k, V_1, \ldots, V_l) \mapsto T^{i_1\ldots i_k}_{j_1\ldots j_l} \omega_{1 i_1} \ldots \omega_{k i_k}, V_1^{j_1} \ldots V_l^{j_l}. \end{aligned}$$

3. A k-form ω is a completely antisymmetric $(0,k)$-tensor i.e.

$$\omega_{i_1 \cdots i_\mu \cdots i_\nu \cdots i_k} = -\omega_{i_1 \cdots i_\nu \cdots i_\mu \cdots i_k}$$

Suppose we have a smooth map $\psi : M \to N$ we can also define a corresponding map between the two tangent bundles:

$$d\psi : TM \to TN : v_\gamma \to v_{\psi\gamma}$$

this map is called the jacobian. In every point $p \in M$ the jacobian gives a linear map $(d\psi)_p : T_pM \to T_{\psi(p)}N$. If this map is a an isomorphism one can use the implicit function theorem to find an open neighborhood U of p such that $\psi|_U$ is a diffeomorphism. We can hence conclude that ψ is a cover map if $(d\psi)_p$ is an isomorphism for every $p \in M$.

Another construction we can make from a map between manifolds is the pullback. This is a map in the opposite direction: it maps the one-forms on N to one-forms on M:

$$\psi^* : \Omega_1(N) \to \Omega_1(M) : \omega \mapsto \psi^*\omega \text{ such that } (\psi^*\omega)_p(V_p) = \omega_{\psi(p)}((d\psi)_p V_p)$$

This trick does not work for vector fields, however if ψ is a cover map we can construct a pull back for vector fields.

$$\psi^* : \mathsf{Vect}(N) \to \mathsf{Vect}(M) : V \mapsto \psi^* V \text{ such that } (\psi^* V)_p = (d\psi)_p^{-1} V_{\psi(p)}$$

Using the tensor product we can extend this product to all possible kinds of tensor fields: $\psi^* : \mathsf{Ten}^{k,l}(N) \to \mathsf{Ten}^{k,l}(M)$

2.2 Metrics and Riemannian Manifolds

A $0,2$-tensor g for which in every point p the map $g_p : T_pM \times T_pM \to \mathbb{R}$ is nondegenerate, symmetric and positive definite, is called a metric. A manifold equipped with a metric g is called a Riemannian manifold. It is costume to write $\langle X, Y\rangle$ as a shorthand for $g(X, Y)$ and $\|X\|$ for $\sqrt{g(X, X)}$.

Lemma 2.4. *Every compact manifold M admits a Riemannian metric.*

Proof. We take a finite cover of charts $\phi_i : U_i \to M$ on M such that their images are the unit open ball $B(0, 1) \subset \mathbb{R}^n$. Now consider the functions

$$\rho_i : M \to \mathbb{R} : x \mapsto \begin{cases} 0 & x \notin U_i \\ e^{-\frac{1}{1-\|\phi_i(x)\|^2}} & x \notin U_i \end{cases}$$

One can easily check that these maps are smooth, strictly positive inside U_i and zero outside U_i. Now for every U_i we can also construct a metric $g^{(i)}$ using the standard euclidean metric on $B(0, 1)$. The sum

$$g = \rho_i g^{(i)}$$

gives a metric on M: it is easy to see that it is positive definite and symmetric, the tricky part is to show that g is smooth, but this follows from the fact that the derivatives in all orders of the functions ρ_i at the boundaries of the U_i are all zero. □

Metrics allow us to define distances of curves: let p be a path. The length of the path p between $p(0)$ and $p(1)$ can then be calculated as

$$\int_0^1 \|\gamma'(t)\| dt.$$

Metrics also allow us to convert vector fields into one-forms and vice versa. If g is a metric we denote its coefficients by $g_{\mu\nu}$. The inverse of this coeffient matrix will be denoted by $g^{\mu\nu}$. These form the coeffients of a $(2,0)$-tensor and we have of course

$$g^{\mu\nu} g_{\nu\kappa} = \delta^\mu_\kappa$$

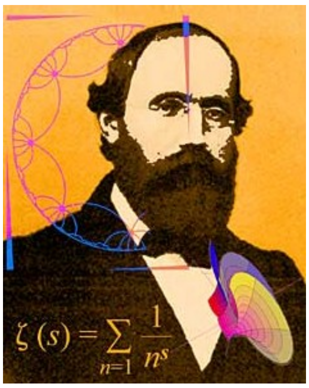
where δ is the Kronecker delta. By contracting with $g_{\mu\nu}$ we can convert a vector X^κ into a one-form $g_{\mu\kappa} X^\kappa$ and a one-form ω_κ into a vector $g^{\mu\kappa}\omega_\kappa$. This process is called lowering/uppering an index and it can also be used to convert (k,l)-tensors into $(k-1,l+1)$- or $(k+1,l-1)$-tensors.

The procedure of uppering and lowering together with contractions gives us a way to define inner products on all tensor fields:

$$\langle , \rangle : T^{k,l}(M) \times T^{k,l}(M) \to \mathbb{C}^\infty : (T,U) \mapsto T^{i_1 \ldots i_k}_{j_1 \ldots j_l} U^{r_1 \ldots r_k}_{s_1 \ldots s_l} g_{i_1 r_1} \ldots g_{i_k r_k} g^{j_1 s_1} \ldots g^{j_l s_l}$$

These maps are all positive definite and bilinear in every point and give us a way to determine the size of a tensor or a form.

Miniature 2: Bernhard Riemann (1826 - 1866)



Riemann was arguably the most influential mathematician of the middle of the nineteenth century. His published works are a small volume only, but opened up research areas combining analysis with geometry. In 1853 Gauss asked his student Riemann to prepare a Habilitationsschrift on the foundations of geometry. Over many months, Riemann developed his theory of higher dimensions. When he finally delivered his lecture in 1854, the mathematical public received it with enthusiasm. The subject founded by this work is Riemannian geometry and the fundamental object is what is now called the Riemann curvature tensor.

Chapter 3

Connections and Curvature

3.1 Connections

Although there is a natural way to differentiate a smooth function defined on a manifold with respect to a tangent vector, there is no natural way to differentiate vector fields. In fact, there are lot of possible rules for differentiating vector fields with respect to a tangent vector, and to choose one of them, (the most appropriate one), the differentiable manifold structure alone is not enough. A fixed rule for the differentiation of vector fields is itself an additional structure on the manifold, called an affine connection.

As far as only vector fields on an open domain of $\mathbb{R}^n$ are considered, the following definition seems to be quite natural.

The derivative of a smooth vector field X on an open subset $U \subset \mathbb{R}^n$ with respect to a tangent vector $Y \in T_p\mathbb{R}^n$ is defined by

$$\nabla_Y X = (X \circ \gamma)(0) \in T_p\mathbb{R}^n,$$

where $\gamma : [-\epsilon, \epsilon] \to U$ is any smooth curve such that $\gamma(0) = p$ and $\gamma(0) = Y$. We see that

$$\nabla_Y X = \sum_{i=1}^{n} Y(X_i)\partial_i(p),$$

where ∂_i denotes the i-th coordinate vector field on $\mathbb{R}^n$, X_i are the components of the vector field X. In particular, the value of $\nabla_Y X$ does not depend on the choice of γ. Note that we can also take a vector field for Y so that we can construct a vector field $\nabla_Y X : p \mapsto \nabla_{Y(p)} X$.

It is easy to check that differentiation of vector fields has the following properties.

(1) $\nabla_{Y_1+Y_2}X = \nabla_{Y_1}X + \nabla_{Y_2}X$
(2) $\nabla_{fY}X = f\nabla_Y X$
(3) $\nabla_Y(X_1 + X_2) = \nabla_Y X_1 + \nabla_Y X_2$
(4) $\nabla_Y(fX) = Y(f)X + f\nabla_Y X$
(5) $\nabla_{X_1}X_2 - \nabla_{X_2}X_1 = [X_1, X_2]$
(6) $Y(\langle X_1, X_2\rangle) = \langle\nabla_Y X_1, X_2\rangle + \langle X_1, \nabla_Y X_2\rangle$

where $X_1, X_2 \in \mathsf{Vect}(\mathbb{R}^n)$, $Y \in T_p\mathbb{R}^n$, $f \in C^\infty(\mathbb{R}^n)$ and $\langle X, Y\rangle = \sum_i X^i Y^i$ is the standard inner product on $\mathbb{R}^n$.

Now we shall study the general case. Let M be a smooth manifold. As we mentioned, there is no natural rule for derivation of vector fields on M, so we introduce such rules axiomatically, as operations satisfying some of the properties (1-6).

Definition 3.1. An affine connection (or briefly a connection) on M is a mapping which assigns to two smooth vector fields Y and X a new one $\nabla_Y X$ called the covariant derivative of the vector field X with respect to the vector field Y, having the first 4 properties above.

The presence of an affine connection on a manifold allows us to differentiate not only vector fields, but also tensor fields of any type. If ω is a 1-form then we define

$$(\nabla_X\omega)(Y) := X(\omega(Y)) - \omega(\nabla_X(Y)).$$

This is indeed again a one form because

$$(\nabla_X\omega)(fY) = f(\nabla_X\omega)(Y).$$

We can extend this to general forms using the Leibniz rule

$$\nabla_X(T_1 \otimes T_2) = \nabla_X T_1 \otimes T_2 + T_1 \otimes \nabla_X T_2.$$

In general if $T : \Omega_1(M)^k \times \mathsf{Vect}(M)^l \to C^\infty(M)$ is a k, l-tensor then

$$\begin{aligned}(\nabla_X T)&(\omega_1, \dots, \omega_k, Y_1, \dots, Y_l)\\ &= X(T(\omega_1, \dots, \omega_k, Y_1, \dots, Y_l))\\ &- T(\nabla_X\omega_1, \dots, \omega_k, Y_1, \dots, Y_l) - \cdots - T(\omega_1, \dots, \omega_k, Y_1, \dots, \nabla_X Y_l).\end{aligned}$$

Let $x_1, \dots, x_n$ be local coordinates defined from a chart on an open subset U of M and $\partial_1, \dots \partial_n$ be the corresponding basis vector fields on U. Given an affine connection ∇ on M, we can express the vector field $\nabla_{\partial_i}\partial_j$ as a linear combination of the basis vector fields

$$\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k \partial_k.$$

The components Γ_{ij}^k are smooth functions called Christoffel symbols. Beware! Although one might expect it from the indexed notation the Γ_{ij}^k are not the coefficients of a $(1,2)$-tensor. However if one has two connections ∇ and ∇' then the difference of their Christoffel symbols is indeed a tensor.

In the index notation we will also use ∇_i as a shorthand for ∇_{∂_i}.

The restriction of an affine connection onto an open coordinate neighborhood U is uniquely determined by the Christoffel symbols. This is because we can write out the expression in its coordinates using the Christoffel symbols.

$$\nabla_Y X = \sum_k (Y(X_k) + \sum_{i,j} X_i Y_j \Gamma_{ij}^k)\partial_k.$$

Observe, that in fact, the tangent vector $(\nabla_Y X)(p)$ depends only on the vector $Y(p)$. Furthermore, we do not need to know the vector field X everywhere on U to compute $(\nabla_Y X)(p)$. It is enough to know X at the points of a curve $\gamma[-\epsilon, \epsilon] \to M$ such that $\gamma(0) = p$, $\gamma(0) = Y(p)$.

Therefore it makes sense to define

Definition 3.2. Let $\gamma : [a, b] \to M$ be a smooth curve in M and take a vector field G such that $G(\gamma(t)) = \gamma'(t)$.

A vector field X is said to be parallel along γ if

$$\nabla_{\gamma'(t)} X := (\nabla_G X)_{\gamma(t)} = 0 \quad \forall t \in [a, b].$$

The curve γ is called a geodesic if

$$\nabla_{\gamma'(t)}\gamma'(t) := \nabla_G G = 0 \quad \forall t \in [a, b].$$

Because of the remarks above the definitions do not depend on the choice of G.

Lemma 3.3. *Given a curve $\gamma : [0, 1] \to M$ and a tangent vector X_0 at the point $\gamma(0)$ then for every $t \in [0, 1]$ there is a unique vector $X_t \in T_{\gamma(t)}M$ such that*

$$\forall t \in [0, 1] : \nabla_{\gamma'(t)} X_t = 0.$$

Proof. If one writes out the condition $\nabla_\gamma X_t = 0$ in local coordinates we get a first order differential equation. Such an equation has a unique solution such that $X_t|_{t=0} = X_0$. □

The vector X_t is said to be obtained from X_0 by parallel transport along γ. In this way we have an isomorphism between the two tangent spaces:

$$\Pi_{(\gamma,t)} : T_{\gamma(0)}M \to T_{\gamma(t)}M$$

It is easy to check that

$$\nabla_{\gamma(0)'}X = \lim_{\epsilon\to 0} \frac{\Pi_{(\gamma,-\epsilon)}X_{\gamma(\epsilon)} - X_{\gamma(0)}}{\epsilon}$$

So the covariant derivation is the ordinary derivation where we use parallel transport to identify the different tangent spaces.

Definition 3.4. A connection is called symmetric or torsion free if it satisfies the identity

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Applying this identity to the case $X = \partial_i, Y = \partial_j$, since $[\partial_i, \partial_j] = 0$ one obtains the relation

$$\Gamma^k_{ij} = \Gamma^k_{ji}$$

The converse also holds: if $\Gamma^k_{ij} = \Gamma^k_{ji}$ then ∇ is torsion-free.

Roughly speaking, the torsion free condition halves the degree of freedom in the choice of Christoffel symbols, a symmetric connection is uniquely determined by $n\frac{n(n+1)}{2}$ functions, nevertheless, the space of symmetric affine connections on a manifold is still infinite dimensional. To reduce further the degree of freedom putting condition on the connection. We can have to introduce a metric on the manifold.

Definition 3.5. A connection ∇ on M is compatible with the Riemannian metric for any vector fields X, Y, Z

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle X, \nabla_Y Z\rangle.$$

or if g is $(0, 2)$-tensor corresponding to the metric then $\nabla_X g = 0$ for every vector field X.

This implies that parallel translation preserves inner products. In other words, for any curve γ and any pair $X_0, Y_0 \in T_{\gamma(0)}M$ The parallel transported vectors X_t, Y_t have thew same inner product.

$$\langle X_0, Y_0\rangle = \langle X_t, Y_t\rangle$$

Theorem 3.6 (Fundamental theorem of Riemannian geometry)**.** *A Riemannian manifold possesses a unique torsion-free connection which is compatible with its metric.*

Proof. Applying the compatibility condition to the basis vector fields ∂_i corresponding to a fixed chart on the manifold and setting $\langle \partial_i, \partial_j >= g$ one obtains the identity

$$\partial_i g_{jk} = \langle \nabla_i \partial_j, \partial_k \rangle + \langle \partial_j, \nabla_i \partial_k \rangle$$

permuting i, j and k this gives three linear equations relating the three quantities

$$\langle \nabla_i \partial_j, \partial_k \rangle,\ \langle \nabla_j \partial_k, \partial_i \rangle,\ \langle \nabla_k \partial_i, \partial_j \rangle$$

(There are only three such quantities since ∇ is torison-free). These equations can be solved uniquely; yielding the first Christoffel identity

$$\langle \nabla_i \partial_j, \partial_k \rangle = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$$

The left hand side of this identity is equal to $\Gamma^l_{ij} g_{lk}$. Multiplying by the inverse (g^{kl}) of the matrix (g_{lk}) this yields the second Christoffel identity

$$\Gamma^l_{ij} = \sum_k \frac{1}{2} g^{kl} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$$

Thus the connection is uniquely determined by the metric. Conversely, defining Γ^l_{ij} by this formula, one can verify that the resulting connection is symmetric and compatible with the metric. □

The unique symmetric affine connection which is compatible with the metric on a Riemannian manifold is called the *Levi-Civita* connection.

The connection ∇ we introduced on open subsets of R is just the Levi-Civita connection of $\mathbb{R}^n$.

Miniature 3: Tullio Levi-Civita (1873 - 1941)

Tullio Levi-Civita was an Italian mathematician, most famous for his work on absolute differential calculus and its applications to the theory of relativity. He was a pupil of Gregorio Ricci-Curbastro, the inventor of the tensor calculus who gave his name to the Ricci tensor. His work included foundational papers in both pure and applied mathematics, celestial mechanics (notably on the three-body problem) and hydrodynamics.
Later on, when asked what he liked best about Italy, Einstein said "spaghetti and Levi-Civita"

3.2 Curvature

If ∇ is an affine connection on a manifold M, then we may consider the operator

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} : \mathsf{Vect}(M) \to \mathsf{Vect}$$

where $[\nabla_X, \nabla_Y]$ is the usual commutator of operators. The mapping that assigns to the vector fields (X,Y) the operator $R(X,Y)$ is called the curvature operator of the connection. The assignment

$$\mathsf{Vect}(M)^3 \to \mathsf{Vect}(M) : (X,Y,Z) \to \mathbb{R}(X,Y)(Z)$$

is called the curvature tensor of the connection. To reduce the number of brackets, we shall denote $R(X,Y)(Z)$ simply by $R(X,Y;Z)$. Thus, the letter R is used in two different meanings, later it will denote also a third mapping, but the number of arguments of R makes always clear which meaning is considered.

Lemma 3.7. *The curvature tensor is linear over the ring of smooth functions in each of its arguments, and it is skew symmetric in the first two arguments.*

Proof. Skew symmetry in the first two arguments is clear, the linearity:

$$R(X, f_1Y_1 + f_2Y_2; Z) = f_1R(X,Y_1;Z) + f_2R(X,Y_2;Z)$$
$$R(X,Y; f_1Z_1 + f_2Z_2) = f_1R(X,Y;Z_1) + f_2R(X,Y;Z_2)$$

can be calculated using the defining properties of a connection. □

The lemma is a bit surprising, because the curvature tensor is built up from covariant derivations, which are not linear operators over the ring of smooth functions. So in every point of M R gives us a linear map

$$R_p : T_pM^3 \to T_pM,$$

so the curvature tensor is in fact a $(1,3)$-tensor. Therefore we denote the coefficients of R by R^ℓ_{ijk}. An easy but tedious calculation gives us the local expression for these coefficients.

$$R^\ell{}_{ijk} = \frac{\partial}{\partial x^j}\Gamma^\ell_{ik} - \frac{\partial}{\partial x^k}\Gamma^\ell_{ij} + \Gamma^\ell_{js}\Gamma^s_{ik} - \Gamma^\ell_{ks}\Gamma^s_{ij}$$

Beside skew-symmetry in the first two arguments, the curvature tensor has many other symmetry properties.

Theorem 3.8 (First Bianchi Identity)**.** *If R is the curvature tensor of a torsion free connection, then R(X,Y;Z) + R(Y,Z;X) + R(Z,X;Y) = 0 for any three vector fields X,Y,Z.*

Proof. Let us introduce the following notation. If $F(X, Y, Z)$ is a function of the vector fields X, Y, Z, then denote by $\circlearrowright F(X, Y, Z)$ the sum of the values of F at all cyclic permutations of the variables (X, Y, Z):

$$\circlearrowright F(X,Y,Z) = \circlearrowright F(Y,Z,X) = \circlearrowright F(Z,X,Y) = F(X,Y,Z)+F(Y,Z,X)+F(Z,X,Y).$$

The theorem claims vanishing of

$$\begin{aligned}\circlearrowright R(X,Y;Z) &= \circlearrowright(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z)\\ &= \circlearrowright(\nabla_Z\nabla_X Y - \nabla_Z\nabla_Y X - \nabla_{[X,Y]}Z)\\ &= \circlearrowright(\nabla_Z[X,Y] - \nabla_{[X,Y]}Z)\\ &= \circlearrowright([Z,[X,Y]]) = 0.\end{aligned}$$

The first equality follows from the cyclic property, the second and the third from the torsion-freeness of ∇ and the last equality is the Jacobi Identity. □

Theorem 3.9 (Second Bianchi Identity)**.** *The curvature tensor of a torsion free connection satisfies the following identity*

$$\begin{aligned}\underset{XYZ}{\circlearrowright}(\nabla_X R)(Y,Z;W) := \underset{XYZ}{\circlearrowright}(XR(Y,Z;W) &- R(\nabla_X Y, Z; W)\\ &- R(Y,\nabla_X Z;W) - R(Y,Z;\nabla_X W))\end{aligned}$$

Proof. After some calculational gymnastics one can rewrite this expression as

$$\underset{XYZ}{\circlearrowright}\nabla_{[[X,Y],Z]}W$$

which is zero because of the Jacobi Identity. □

The geometric interpretation of R is in terms of parallel transport. If X_p, Y_p, Z_p are tangent vectors in p (and X_p is linearly independent of Y_p) we can construct a chart ϕ such that $\partial_1(p) = X_p$ and $\partial_2(p) = Y_p$. Now consider a little loop γ in M that starts in p then goes along the coordinate lines through the points with coordinates $(x_1, x_2) = (\epsilon, 0), (\epsilon, \epsilon), (0, \epsilon), (0, 0) = p$. We can consider the parallel transport of Z_p around this loop. The difference of Z_p and its parallel transport depends on ϵ^2 so we can look at

$$\lim_{\epsilon\to 0}\frac{\Pi_\gamma(Z_p) - Z_p}{\epsilon^2}$$

This limit is equal to $R(X_p, Y_p; Z_p)$, so the curvature measures the infinitesimal parallel transport around a loop in p.

If $(M, \langle,\rangle)$ is a Riemannian manifold with Levi-Civita connection ∇, and R is the curvature tensor of ∇, then we can introduce a tensor R of valency $(0, 4)$, related to R by the equation

$$R(X,Y;Z,W) = \langle R(X,Y;Z), W\rangle.$$

is the Riemann-Christoffel curvature tensor of the Riemannian manifold.

To simplify notation, we shall denote R also by R. This will not lead to confusion, since the Riemann-Christoffel tensor and the ordinary curvature tensor have different number of arguments. Levi-Civita connections are connections of special type, so it is not surprising, that the curvature tensor of a Riemannian manifold has stronger symmetries than that of an arbitrary connection. Of course, the general results can be applied to Riemannian manifolds as well, and yield

$$R(X,Y;Z,W) = -R(Y,X;Z,W), \quad \underset{XYZ}{\circlearrowright} R(X,Y;Z,W) = 0.$$

In addition to these symmetries, we have the following ones.

Theorem 3.10. *The Riemann-Christoffel curvature tensor is skew-symmetric in the last two arguments end symmetric in swapping the first two with the last two arguments*

$$\begin{aligned} R(X,Y;Z,W) &= -R(X,Y;W,Z) \\ &= R(Z,W;X,Y) \end{aligned}$$

Proof. The first one can be established by writing it out and using the compatibility of the connection and the metric. The second involves summing several different permutations of the second bianchi identity. □

Because of these identities we can consider R also as a symmetrics map bilinear

$$R : \Omega_2 M \times \Omega_2 M \to \mathbb{C}^\infty(M) : (X \wedge Y, Z \wedge W) \mapsto R(X,Y;W,Z).$$

As is known from linear algebra a symmetric bilinear map on a vector space is completely determined by its quadratic form. The metric on M gives us a way to determine the lenght of forms $\Omega_2(M)$ using the definition

$$\|X \wedge Y\|^2 = \langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2$$

therefore we define

Definition 3.11. The sectional curvature is a map

$$K := \Omega_2(M) \to \mathbb{C}^\infty : X \wedge Y \to K(X,Y) = \frac{R(X,Y;Y,X)}{\|X \wedge Y\|^2}$$

This map is invariant under rescaling $K(fX \wedge Y) = K(X \wedge Y)$ and hence nonlinear. Moreover $K(X,Y)_p$ only depends on the plane spanned by X_p, Y_p. We will see in chapter 5 that the geometric interpretation of K is that K measures the rate of change of the sum of the angles of a triangle in terms of its size.

Definition 3.12. Riemannian manifolds, the sectional curvature function of which is constant i.e. does not depend on p, X_p, Y_p, are called spaces of constant curvature. The space is elliptic or spherical if $K > 0$, K is parabolic or Euclidean if $K = 0$ and is hyperbolic if $K < 0$.

In the next chapter we shall study these spaces in more detail.

The curvature tensor is a complicated object containing a lot of information about the geometry of the manifold. There are some useful ways to derive some simpler tensor fields from the curvature tensor. Of course, the simplification is paid by losing information.

Definition 3.13. Let (M, ∇) be a manifold with an affine connection, R be the curvature tensor of ∇. The Ricci tensor Ric of the connection is a $(0, 2)$-tensor obtained by contraction

$$\mathrm{Ric}_{\mu\nu} := R^{\lambda}_{\mu\lambda\nu}$$

From the symmetry properties of R we can easily deduce that Ric is a symmetric tensor: $\mathrm{Ric}_{\mu\nu} = \mathrm{Ric}_{\nu\mu}$.

Using the metric we can do one more contraction:

$$\mathrm{R} := g^{\mu\nu}\mathrm{Ric}_{\mu\nu}$$

This gives use a smooth function on M which is called the scalar curvature.

3.3 Geodesics

We have defined the length of a smooth curve $\gamma : [a, b] \to M$ as

$$\|\gamma\| = \int_a^b \sqrt{\langle\gamma'(t), \gamma'(t)\rangle}dt.$$

in this definition γ does not need to be completely smooth, it can also be piecewise smooth: γ is continuous and smooth in all but a finite number of $t_i \in [a, b]$.

The metric of manifold now allows us to define a distance function between two points as the length of the shortest piecewise smooth path from p to q

$$d(p, q) := \inf_{\gamma : p \to q} \|\gamma\|$$

It is easy to show that this is a distance function (symmetric, $d(p, q) = 0 \Leftrightarrow p = q$, triangle inequality).

To find the analog of straight lines in the intrinsic geometry of a Riemannian manifold we have to characterize straight lines in a way that makes sense for Riemannian manifolds as well. Since the length of curves is one of the most fundamental concepts of Riemannian geometry, we can take the following characterization: a curve is a straight line if and only if for any two points on the curve, the segment of the curve bounded by the points is the shortest among curves joining the two points.

This characterization does not depend on the parametrization of the curve. Therefore we add an extra condition: uniform parametrization. We say a curve is uniformly parametrized if $\|\gamma'(t)\|$ is constant. Every curve can be uniformly parametrized.

Theorem 3.14. *If γ is a uniformly parametrized curve of minimal length between p, q then γ is a geodesic.*

Proof. A deformation of a curve $\gamma : [a, b] \to M$ is a smooth map $\tilde{\gamma} : [-\delta, \delta] \times [a, b] \to M : (s, t) \to \tilde{\gamma}_s(t)$ such that $\tilde{\gamma}_0 = \gamma$ and γ_s is a path from p to q for every s. This gives us two vector fields on the image of $\tilde{\gamma}$: $\frac{d}{ds}$ and $\frac{d}{dt}$, moreover one can see easily that $[\frac{d}{ds}, \frac{d}{dt}] = 0$ Because γ is uniformly parametrized, $\|\frac{d}{dt}\|_{s=0}$ is a constant ℓ.

The curves γ_s all have length at least $\|\gamma\|$ so

$$\frac{d}{ds}\|\gamma_s\|_{s=0} = 0$$

We can write this out

$$\begin{aligned}
\int_a^b \frac{d}{ds}\sqrt{\langle \frac{d}{dt}, \frac{d}{dt}\rangle}|_{s=0}dt &= \int_a^b \frac{\frac{d}{ds}\langle \frac{d}{dt}, \frac{d}{dt}\rangle}{2\|\frac{d}{dt}\|}|_{s=0}dt \\
&= \int_a^b \frac{\langle \nabla_{\frac{d}{ds}}\frac{d}{dt}, \frac{d}{dt}\rangle + \langle \frac{d}{dt}, \nabla_{\frac{d}{ds}}\frac{d}{dt}\rangle}{2\|\frac{d}{dt}\|}|_{s=0}dt \\
&= \frac{1}{\ell}\int_a^b \langle \nabla_{\frac{d}{ds}}\frac{d}{dt}, \frac{d}{dt}\rangle|_{s=0}dt \\
&= \frac{1}{\ell}\int_a^b \langle \nabla_{\frac{d}{dt}}\frac{d}{ds} + \overbrace{[\frac{d}{ds}, \frac{d}{dt}]}^{=0}, \frac{d}{dt}\rangle|_{s=0}dt \\
&= \frac{1}{\ell}\int_a^b \frac{d}{dt}\langle \frac{d}{ds}, \frac{d}{dt}\rangle - \langle \frac{d}{ds}, \nabla_{\frac{d}{dt}}\frac{d}{dt}\rangle|_{s=0}dt \\
&= \frac{1}{\ell}[\langle \frac{d}{ds}, \frac{d}{dt}\rangle|_{s=0}]_a^b - \frac{1}{\ell}\int_a^b \langle \frac{d}{ds}, \nabla_{\frac{d}{dt}}\frac{d}{dt}\rangle|_{s=0}dt \\
&= -\frac{1}{\ell}\int_a^b \langle \frac{d}{ds}, \nabla_{\frac{d}{dt}}\frac{d}{dt}\rangle|_{s=0}dt
\end{aligned}$$

As the last formula must be zero for all possible deformations $\tilde{\gamma}$ it must hold that $\nabla_{\frac{d}{dt}}\frac{d}{dt}|_{s=0} = 0$. This means that

$$\nabla_{\gamma'(t)}\gamma'(t) = 0$$

or γ is a geodesic. $\square$

remark 1. The converse however is not true: there exist geodesic curves that do not minimize the length, f.i. closed geodesics ($\gamma(a) = \gamma(b)$).

One can prove that any geodesic is locally minimizing its lenght i.e. $\forall t \in [a, b]$: $\exists [a_t, b_t] \ni t$ such that $\gamma_{[a_t,b_t]}$ is a path of minimal length from $\gamma(a_t)$ to $\gamma(b_t)$.

If γ is a curve and γ^k are the coordinates of of γ in a local chart, then we can write the geodesic condition as

$$\frac{d^2}{dt^2}\gamma^k + \Gamma^k_{ij}(\gamma)\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0$$

This is a second order differential equation. Given initial values $\gamma^k(0)$ and $\gamma^{k\prime}(0)$ there exist a unique solution to this equations. Therefore we can conclude that

Lemma 3.15. *Given a point $p \in M$ and a vector $X \in T_pM$ there exist a unique maximal geodesic $\gamma :]a, b[\to M$ such that $\gamma(0) = P$ and $\gamma'(0) = X$. (the maximality of the geodesic must by understood as the fact that $]a, b[$ is the biggest interval on which γ can be defined.*

Definition 3.16. We will call a Riemannian manifold geodesically complete if all maximal geodesics $\gamma_{p,X}$ are defined on whole $\mathbb{R}$.

If this is the case we define the exponential map as

$$\exp_p(X) := \gamma_{p,X}(1).$$

This maps in every point the tangent space T_pM to M.

Lemma 3.17. *If M is path connected and geodesically complete then $\exp_p$ is a surjective map.*

Proof. Suppose q is a point in M and $\gamma : [0, 1] \to M$ is a path of minimal length between p and q then γ is a geodesic and $\gamma(t) = \exp_p(t\gamma'(0))$. so $q = \exp_p(\gamma'(0))$. $\square$

Lemma 3.18. *For every point p we can find a neighborhood U of $0 \in T_pM$ such that $\exp_p|_U$ is one to one.*

Proof. We can prove this using the implicit function theorem. Therefore we have to prove that the jacobian of the map $\exp_p$ in 0 is an invertible map. But this jacobian is the identity map because $T_0(T_pM) \cong T_pM$ and $\frac{d}{dt}\exp_p(tX)|_{t=0} = X$. □

It is important to note that $\exp_p$ is not always a cover map, one can only find a neighborhood of the zero point in T_pM for which exp is invertible. For other points in T_pM this might not be true.

The metric on M gives us a positive definite symmetric bilinear form on T_pM and we can chose an orthonormal basis for this form. This basis gives us an identification of $\iota : T_pM \to \mathbb{R}^n$ and hence $\iota \circ \exp_p|_U^{-1}$ is a chart on M. A chart of this form is called a normal chart, the corresponding coordinates are also called normal.

Lemma 3.19. *If $x_1, \ldots, x_n$ are normal coordinates around p then we have the following equations in p:*

- $\Gamma^k_{ij}(p) = 0$,
- $g_{ij}(p) = \delta_{ij}$ *and* $\partial_i g_{jk} = 0$,
- $\partial_i\partial_j g_{kl}(p) = \frac{1}{3}R_{iklj}(p)$

These equations only hold in the origin of the coordinate system, in other points they are not valid.

3.4 Isometric maps

Definition 3.20. A locally isometric map $\phi : M \to N$ is a smooth map such that for every p the map $d\phi_p : T_pM \to T_{\phi(p)}N$ is an isomorphism that preserves the metric.

The definition implies that locally isometric maps are covers. On the other hand if $\phi : M \to N$ is a cover and N is a Riemannian manifold than we can define a unique metric on M such that ϕ is an isometric map:

$$g_M(X_p, Y_p) := g_N(d\phi(X_p), d\phi(Y_p)).$$

this metric is the pullback of the metric on M.

Theorem 3.21. *Two isometric maps $\phi_1, \phi_2 : M \to N$ such that there exists a point p with $\phi_1(p) = \phi_2(p)$ and $(d\phi_1)_p = (d\phi_2)_p$ are equal.*

Proof. Because isometric maps map geodesic to geodesics, they commute with the exponential maps i.e.

$$\phi_1 \exp_p(X) = \exp_{\phi_1(p)}(d\phi_1(X)).$$

Now let $q \in M$ and let $X \in T_pM$ be such that $\exp_p(X) = q$ then

$$\phi_2(q) = \phi_2 \exp_p(X) = \exp_{\phi_2(p)}(d\phi_2(X)) = \exp_{\phi_1(p)}(d\phi_1(X)) = \phi_1 \exp_p(X) = \phi_1(q).$$

□

Since the uniques of the isometric map is now established, we would also like to have a criterion for the existence of it.

Definition 3.22. A unwinding on two Riemannian manifolds is a triple $(\gamma^\wedge, \gamma^\vee, \Phi)$ consisting of 2 paths $\gamma^\wedge M : [0,1] \to M$, $\gamma^\vee : [0,1] \to N$ and a sequence of orthonormal isomorphisms $\Phi(t) : T_{\gamma^\wedge}M \to T_{\gamma^\vee}N$ that commute with the parallel transport and is compatible with tangent vectors:

$$\begin{aligned} \forall t \in [0,1] : &\Phi_t \circ \Pi_{t,\gamma^\wedge} = \Pi_{t,\gamma^\vee} \circ \Phi(0) \\ &\Phi_t({\gamma^\wedge}'(t)) = {\gamma^\vee}'(t) \end{aligned}$$

The curve $\gamma^\wedge$ is called the upper curve the curve $\gamma^\vee$ is the lower curve.

From the definition we immediately can deduce that Φ is uniquely determined by $\Phi(0)$. A bit more trickier to see is that the upper (lower) curve and $\Phi(0)$ completely determine the lower (upper) curve: this follows from the fact that the conditions can be expressed as a system of linear differential equations which has a unique solution if M and N are complete.

If there exists a locally isometric map ϕ between M and N we can construct unwindings in the a more easy way:

$$(\gamma, \phi\gamma, (d\phi)_{\gamma t}).$$

All these unwindings have the property that the endpoint of $\phi\gamma$ is uniquely determined by the endpoint of γ not by the form of γ.

We can ask ourselves whether the converse holds as well. This is indeed true if M is complete and connected.

Theorem 3.23. *Suppose M is complete and connected. Let $p \in M$ and $q \in N$ and let $\Psi : T_pM \to T_qN$ be an orthonormal isomorphism. If for all unwindings $(\gamma_i^\wedge, \gamma_i^\vee, \Phi_i)$ with $(\Phi_i)(0) = \Psi$ holds that*

$$\gamma_i^\wedge(1) = \gamma_j^\wedge(1) \Rightarrow \gamma_i^\vee(1) = \gamma_i^\vee(1),$$

then there exists a unique isometric map $\phi : M \to N$ with $(d\phi)_p = \Psi$.

Proof. We can define the map by the equation

$$\phi(\gamma^\wedge(1)) = \gamma^\vee(1)$$

for every $(\gamma^\wedge, \gamma^\vee, \Phi)$ with $\Phi(0) = \Psi$. This map is surjective because if we choose a path $\gamma^\vee$ from q to $r \in N$ we can always find an unwinding of the form $(-, \gamma^\vee, -)$.

It is also a smooth isometric map because we can check that $\|d\phi(\gamma^{\wedge\prime}(1))\| = \|(\phi\gamma^\wedge)'(1)\| = \|\gamma^\vee(1)\| \neq 0$ if $\gamma^{\wedge\prime}(1) \neq 0$. □

If M is simply connected we can go even further:

Theorem 3.24 (Cartan-Ambrose-Hicks)**.** *Suppose M is complete and simply connected. Let $p \in M$ and $q \in N$ and let $\Psi : T_pM \to T_qN$ be an orthonormal isomorphism. If for all unwindings $(\gamma^\wedge, \gamma^\vee, \Phi)$*

$$\Phi(t)^* {R^\vee}_{\gamma^\vee(t)} = {R^\wedge}_{\gamma^\wedge(t)}$$

(i.e. the pullback of the curvature on N is the curvature on M) then there exists a unique isometric map $\phi : M \to N$ with $(d\phi)_p = \Psi$.

Proof. With the previous theorem in mind we just have to prove that the endpoints of the lower curves of two unwindings coincide if the endpoints of the upper curves coincide.

Let $\gamma_1^\wedge$ and $\gamma_2^\wedge$ be two such upper curves then because of the simply connectedness of M there exists a homotopy H connecting these curves. We denote the intermediate curves of this homotopy by $\gamma_\lambda^\wedge$ for each $\lambda \in [0, 1]$. Because N is complete we can find for each $\gamma_\lambda^\wedge$ an unwinding $(\gamma_\lambda^\wedge, \gamma_\lambda^\vee, \Phi_\lambda)$.

A homotopy has two parameters the time parameter t and the deformation parameter λ. Nothing prevents us to interchange these parameters so we get a 2 new sets of curves: $\gamma_-^\wedge(t), \gamma_-^\vee(t)$ (one for every value of t). The first thing we are going to prove is that these also form unwindings $(\gamma_-^\wedge(t), \gamma_-^\vee(t), \Phi_-(t))$. To prove this one has to show that

$$\Phi_\lambda(t)\partial_\lambda\gamma_\lambda^\wedge(t) = \partial_\lambda\gamma_\lambda^\vee(t). \ [*]$$

This allows us to conclude that

$$\partial_\lambda \gamma^\vee(1) = \Phi_\lambda(t)\partial_\lambda \gamma^\wedge_\lambda(t) = \Phi_\lambda(t)0 = 0$$

so $\gamma^\vee_0(1) = \gamma^\vee_1(1)$.

We will show equation [∗] this by finding as set of linear differential equations (and initial conditions) that both sides of the equation above satisfy.

To find these we need first some notation. Choose an orthogonal basis $E^\wedge_{1p}, \ldots, E^\wedge_{kp} \in T_pM$ and define $E^\wedge_i(\lambda, t) \in T_{\gamma^\wedge_\lambda(t)}M$ and $E^\vee_i(\lambda, t) \in T_{\gamma^\vee_\lambda(t)}N$ by

$$E^\vee_i(\lambda, t) := \Pi_{\gamma^\wedge_\lambda(t)} E^\wedge_{ip} \text{ and } E^\vee_i(\lambda, t) := \Phi_\lambda(t) E^\wedge_i(\lambda, t).$$

This gives us orthogonal bases for the tangent space in every point of all the curves. We can express the tangent vectors defined by γ_λ in terms of these bases:

$$\partial_t \gamma^\wedge_\lambda(t) =: \sum_i \xi^i(\lambda, t) E^\wedge_i(\lambda, t)$$

Because of the definition of the bases downstairs we also have that

$$\gamma^{\vee\prime}_\lambda(t) =: \sum_i \xi^i(\lambda, t) E^\vee_i(\lambda, t)$$

Now define $X^\vee(\lambda, t) := \partial_\lambda \gamma^\vee_\lambda(t)$ and $Y^\vee_i(\lambda, t) := \nabla_{\partial_\lambda \gamma^\vee} \gamma^\vee_\lambda(t)$. Then we have that

$$\begin{aligned}
\nabla_{\partial_t \gamma^\vee} X^\vee &= \nabla_{\partial_t \gamma^\vee} \partial_{\lambda \gamma^\vee} \\
&= \nabla_{\partial_\lambda \gamma^\vee} \partial_t \gamma^\vee \\
&= (\partial_\lambda \xi^i) E^\vee_i + \xi^i \nabla_{\partial_\lambda \gamma^\vee} E^\vee_i \\
&= (\partial_\lambda \xi^i) E^\vee_i + \xi^i Y^\vee_i
\end{aligned}$$

and

$$\begin{aligned}
\nabla_{\partial_t \gamma^\vee} Y^\vee_i &= \nabla_{\partial_t \gamma^\vee} \nabla_{\partial_\lambda \gamma^\vee} E^\vee_i \\
&= \nabla_{\partial_t \lambda \gamma^\vee} \nabla_{\partial_t \gamma^\vee} E^\vee_i + R^\vee(\partial_t \gamma^\vee, \partial_\lambda \gamma^\vee) E^\vee_i \\
&= R^\vee(\partial_t \gamma^\vee, \partial_\lambda \gamma^\vee, E^\vee_i)
\end{aligned}$$

For the first calculation we used the torsion-freeness of the connection for the second we used the formula for the curvature together with the fact that $[\partial_\lambda \gamma^\vee, \partial_t \gamma^\vee]$

Similarly define $X^\wedge(\lambda, t) := \Phi_\lambda(t)\partial_\lambda \gamma^\wedge_\lambda(t)$ and $Y^\wedge_i(\lambda, t) := \Phi_\lambda(t)\nabla_{\partial_\lambda \gamma^\wedge_\lambda(t)}$. Then we have that

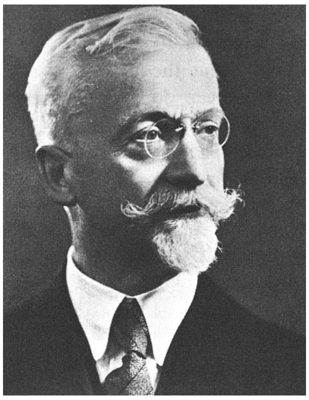
$$\begin{aligned}
\nabla_{\partial_t \gamma^\vee} X^\wedge &= \Phi_\lambda(t) \nabla_{\partial_t \gamma^\wedge} \partial_{\lambda \gamma^\wedge} \\
&= \Phi_\lambda(t)(\partial_\lambda \xi^i) E^\wedge_i + \xi^i \nabla_{\partial_\lambda \gamma^\wedge} E^\wedge_i) \\
&= (\partial_\lambda \xi^i) E^\vee_i + \xi^i Y^\wedge_i
\end{aligned}$$

and

$$\begin{aligned}\nabla_{\partial_t\gamma^\vee} Y_i^\wedge &= \Phi_\lambda(t)(\nabla_{\partial_t\gamma^\wedge}\nabla_{\partial_\lambda\gamma^\wedge} E_i^\wedge) \\ &= \Phi_\lambda(t)(R^\wedge(\partial_t\gamma^\wedge, \partial_\lambda\gamma^\wedge, E_i^\wedge) \\ &= R^\vee(\partial_t\gamma^\vee, \partial_\lambda\gamma^\vee, E_i^\vee)\end{aligned}$$

As $X^\wedge(\lambda, 0) = Y_i^\wedge(\lambda, 0) = X^\vee(\lambda, 0) = Y_i^\vee(\lambda, 0) = 0$ we can conclude that $X^\wedge = X^\vee$ and this finishes the proof. □

Miniature 4: Élie Cartan (1869 - 1951)



Élie Joseph Cartan was an influential French mathematician, who did fundamental work in the theory of Lie groups and their geometric applications. He also made significant contributions to mathematical physics, differential geometry, and group theory.
Cartan was born in Dolomieu in Savoie, and became a student at the École Normale Superieure in Paris in 1888. After his doctorate in 1894, he took lecturing positions in Montpellier and Lyon and Nancy. He died in Paris where he was professor from 1912 and untill he retired in 1942.

Chapter 4

Constant Curvature

4.1 Simply connected models

We recall the definition of the previous chapter.

Riemannian manifolds, the sectional curvature function of which is constant i.e. does not depend on p, X_p, Y_p, *are called spaces of constant curvature. The space is elliptic or spherical if* $K > 0$, K *is parabolic or Euclidean if* $K = 0$ *and is hyperbolic if* $K < 0$.

In this chapter we will show how one can try to classify these manifolds. First we will have a look at the simply connected manifolds because every manifold of constant curvature has a simply connected cover and if we pull back the metric this cover has also has constant curvature.

Theorem 4.1. *Let* M *and* N *be two manifolds of the same constant curvature* K *and fix two points* $p \in M, q \in N$ *and a orthogonal isomorphism* $f : T_pM \to T_qN$. *If* M *is simply connected then there exists a unique locally isometric map* $\phi : M \to N$ *with* $d\phi_p = f$

Proof. First we will give an explicit expression for the curvature in terms of the metric:

$$R_{ijkl} = K(g_{ik}g_{jl} - g_{jk}g_{il}).$$

To prove this we define

$$Q_{ijkl} := R_{ijkl} - K(g_{ik}g_{jl} - g_{jk}g_{il}).$$

then we have of course that Q has the same symmetries as R:

$$\begin{aligned} &(1) \quad Q_{ijkl} = Q_{klij} \\ &(2) \quad Q_{ijkl} = -Q_{jikl} \\ &(3) \quad Q_{ijkl} + Q_{jkil} + Q_{kijl} = 0 \end{aligned}$$

And by the definition of sectional curvature we have

$$Q_{ijkl} v^i w^j v^k w^l = 0$$

substituting v for $u + v$ we get that

$$0 = Q_{ijkl}(v + w)^i w^j (u + v)^k w^l = Q_{ijkl}(u^i v^j v^k w^l + v^i w^j u^k w^l) \overset{(1)}{=} 2Q_{ijkl} u^i w^j v^k w^l.$$

Now substitute w to $w + z$:

$$0 = Q_{ijkl} u^i (w + z)^j v^k (w + z)^l = Q_{ijkl} u^i w^j v^k z^l + Q_{ijkl} u^i z^j v^k w^l.$$

so $Q_{ijkl} = -Q_{ilkj}$ (4). If we combine this with (3) we get

$$0 = Q_{ijkl} + Q_{jkil} + Q_{kijl} \overset{(2),(4)}{=} Q_{ijkl} - Q_{kjil} - Q_{jikl} \overset{(4)(2)}{=} Q_{ijkl} + Q_{ijkl} + Q_{ijkl} = 3Q_{ijkl}.$$

This implies that Q is the zero tensor. So we have shown that if M has constant curvature, the curvature tensor in p does only depend on the value of g in p and not of the derivatives of g. This means that any orthogonal isomorphism $\Phi : T_pM \to T_qN$ between tangent space of two manifolds with the same constant curvature also preserves the curvature tensor: $\Phi^* R_N = R_M$. Therefore all possible unwindings satisfy the condition for the Cartan-Ambrose-Hicks theorem. ☐

Theorem 4.2. *For every $n \geq 2$ and every $K \in \mathbb{R}$ there exists an n-dimensional simply connected complete manifold with constant curvature K. This Riemannian manifold is unique up to isometry.*

Proof. The uniqueness follows from the previous theorem. We only have to construct models for every $K = 0, 1, -1$ and n. For the other K we can rescale the metric: if $\tilde{g}_{ij} = cg_{ij}$ then $\tilde{R}_{ijkl} = cR_{ijkl}$ and $\tilde{K} = c^{-1}K$.

Treat the 3 cases separately.

- $K = 0$. The model here is euclidean space: $M = \mathbb{E}_n = \mathbb{R}^n$ $g_{ij} = \delta_{ij}$. Because the coefficients of the metric are constant the curvature is zero.

  The group of isometries working on the euclidean space is called the Euclidian group $\mathsf{E_n}$ and it consists of all maps of the form

  $$\phi : x \mapsto Ax + b$$

  where A is an orthogonal matrix ($AA^T = 1$) and $b \in \mathbb{R}^n$. It is easy to see that all these maps are isometries and these are all possible maps because $\phi = b$ and $d\phi_0 = A$ uniquely determine the isometry.

- $K = 1$. The model is the n-sphere:

$$M = \mathbb{S}_n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | \sum_i x_i^2 = 1\}$$

The metric comes from the metric in the euclidean space. We can prove easily that $\mathbb{S}_n$ has constant sectional curvature. First look at the group of isometries of the euclidean space (E_{n+1}) that maps the sphere unto itself. This group is

$$\mathsf{O}_{n+1} = \{\phi : x \mapsto Ax | AA^T = 1\}$$

This group works transitive on the points of the sphere and the stabilizer of a point

$$\mathsf{Stab}_p = \mathsf{O}_n$$

works transitive on the two-dimensional subspaces of $T_p\mathbb{S}_n$ and hence all the sectional curvatures must be the same.

If we fix such a plane $\alpha \in T_p\mathbb{S}_n$ we can look at the image of α under $\exp_p$. It is easy to see that this image will be a 2-sphere with radius K^{-1}. So the sectional curvature of α will be the same as the sectional curvature of such a two sphere. Let us calculate this:

We can take spherical coordinates:

$$\begin{aligned} x_0 &= \cos\theta_1, \\ x_1 &= \sin\theta_1 \cos\theta_2 \\ x_2 &= \sin\theta_1 \sin\theta_2 \end{aligned}$$

These coordinates have the advantage that $\langle \partial_{\theta_1}, \partial_{\theta_2} \rangle = 0$ and

$$\begin{aligned} g_{11} &= 1 \\ g_{22} &= \sin^2\theta_1 \end{aligned}$$

The Christoffel symbols are also easy to compute:

$$\begin{aligned} \Gamma^1_{22} &= -\sin\theta_1 \cos\theta_1 \\ \Gamma^1_{12} &= \Gamma^1_{21} = \cot\theta_1 \end{aligned}$$

All other symbols are zero. Finally we compute the curvature:

$$R_{1221} = -\sin^2\theta_1 = (-g_{22}g_{11})$$

Because O_{n+1} works transitive on the points and $\mathsf{Stab}_p \cong \mathsf{O}_n$ we can conclude that the group $\mathsf{S}_n := \mathsf{O}_{n+1}$ contains all isometries of the n-sphere.

- $K = -1$. Take the $n+1$-dimensional Minkowski space. This is the vector space $\mathbb{R}^{n+1}$ equipped with the bilinear form

$$\mu(\vec{x}, \vec{y}) = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i.$$

The symmetry group of this bilinear form is

$$\mathsf{O(1,n)} = \{A \in \mathsf{Mat}_{n+1} | A \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} A^T = 1\}$$

Consider the quadric $-x_0^2 + x_1^2 + \cdots + x_n^2 = -1$. This consists of two disjoint parts $\mathbb{H}^+$ and $\mathbb{H}^-$ containing the points with $x_0 \geq 1$ resp. $x_0 \leq -1$. The group $\mathsf{O(1,n)}$ works transitive on the points of the quadric. We consider the subgroup $\mathsf{O^+(1,n)}$ that maps $\mathbb{H}^+$ to itself.

$$\mathsf{O^+(1,n)} = \{A \in \mathsf{O(1,n)} | A_{00} > 0\}$$

We claim that $\mathbb{H}^+$ equipped with the metric coming from the minkowski bilinear form is a manifold of constant negative curvature.

First of all we show that the metric on $\mathbb{H}^+$ is positive definite. We parametrize $\mathbb{H}^+$ as follows For every point $(x_1, \ldots, x_n) \in \mathbb{D}_n$ we define a point on $\mathbb{H}^n$

$$(\lambda, \lambda x_1, \ldots, \lambda x_n) \text{ with } \lambda = \frac{1}{\sqrt{1 - x_1^2 - \cdots - x_n^2}}$$

This gives us a chart with vector fields

$$\partial_k = \frac{1}{\sqrt{1 - \sum_l x_l^2}^2}(-x_k, -x_k x_1 + \frac{1}{2}\delta_{k1}(1 - \sum_l x_l^2), \ldots, -x_k x_1 + \frac{1}{2}\delta_{kn}(1 - \sum_l x_n^2))$$

Therefore the coefficients of the metric are

$$g_{ij} = \mu(\partial_i, \partial_j) = \frac{\delta_{ij}}{4(1 - \sum_l x_l^2)^2}$$

To prove that $\mathbb{H}_n = \mathbb{H}^+$ has constant negative curvature, we proceed as in the positive case. Again $\mathsf{O^+(1,n)}$ acts transitively on the 3-dimensional subspaces of $\mathbb{R}^{n+1}$ through the origin. Therefore the curvature is constant. To compute the sectional curvature we introduce coordinates We can take spherical coordinates:

$$\begin{aligned} x_0 &= \cosh \theta_1, \\ x_1 &= \sinh \theta_1 \cos \theta_2 \\ x_2 &= \sinh \theta_1 \sin \theta_2 \end{aligned}$$

and we compute the curvature:

$$R_{1221} = \sinh^2\theta_1 = (g_{22}g_{11})$$

The group $\mathsf{H_n} = \mathsf{O}^+(1,\mathsf{n})$ works transitive on the points and the stabilizer of $(1,0,\dots,0)$ is equal to $\mathsf{O_n}$ so we can conclude that this group is the group of isometries of $\mathbb{H}_n$.

□

The classification of complete manifolds with constant curvature can now be stated using the notion of a discrete action

Definition 4.3. If M is a manifold or a topological space then $S \subset M$ is a discrete subset if for every $p \in M$ there exists a neighborhood $U \ni p$ such that $S \cap U$ is finite. If G is a group with an action on M then we call this action discrete if all orbits are discrete.

If $M \to N$ is a cover then the group of deck transformations has a discrete action on M. This is because every $m \in M$ has a small neighborhood.

If $\tilde{M} \to M$ is the universal cover of a complete Riemannian manifold M with constant curvature $0, \pm 1$ then we can pull back the curvature on M to $\tilde{M}$. $\tilde{M}$ is also complete because the geodesics of $\tilde{M}$ are the pull backs of geodesics of M.

Because $\tilde{M}$ is simply connected we know that $\tilde{M} \cong \mathbb{E}_n, \mathbb{S}_n, \mathbb{H}_n$. Also by construction the deck transformations preserve the metric so they form a subgroup $\pi_1(M)$ of $\mathsf{Iso}(\tilde{M}) = \mathsf{E_n}, \mathsf{S_n}, \mathsf{H_n}$. This group is a discrete subset of $\mathsf{Iso}(\tilde{M})$ because the unit $1 \in \pi_1(M)$ (and hence every element) has a neighborhood in $\mathsf{Iso}(\tilde{M})$ that intersects $1 \in \pi_1(M)$ in one point: Take a small neighborhood U_p of $p \in \tilde{M}$ and look at the isometries that map p in U_p. In this open subset there is only one deck transformation: the one that maps p to itself.

Now let M be a simply connected complete manifold. If G is a discrete subgroup of $\mathsf{Iso}M$ and no element of M is fixed by a $g \in \mathsf{G} \setminus \{1\}$, we can construct the quotient manifold

$$M/\mathsf{G} = \{\mathsf{G}p | p \in M\}$$

If we denote by d_p the minimum distance between p and one of its images under G then we can find an open neighborhood $U_p = \{x \in M | d(x,p) < d_p/2\}$ such that $U_p \cap gU_p = \emptyset$ if $g \in \mathsf{G} \setminus \{1\}$. The projection map $M \to M/\mathsf{G}$ restricted to U_p is an injection so we can use a chart on U_p as a chart on M/G. We use these charts to define the smooth structure on M/G.

Now suppose we have two groups $\mathsf{G}_1, \mathsf{G}_2 \subset \mathsf{Iso}M$ such that there quotients are isomorphic as Riemannian manifolds. If $\phi : M/\mathsf{G}_1 \to M/\mathsf{G}_2$ is the corresponding isometry. Choose a couple of point $p, qinM$ such that $\phi(p\mathsf{G}_1) = q\mathsf{G}_2$ The map ϕ gives us an isometric identification $(d\phi)_p : T_pM \to T_qM$. This identification extends to an isometry $\Phi : M \to M$. It is easy to see that

$$\forall x \in M : \Phi(x)\mathsf{G}_2 = \phi(x\mathsf{G}_1),$$

so if $g \in G_1$ then $\Phi g \Phi^{-1} \in \mathsf{G}_2$. The two subgroups are conjugates in $\mathsf{Iso}M$.

Theorem 4.4. *The complete n-dimensional manifolds with constant curvature $0, \pm 1$ up to isometry are in one to one correspondence with the conjugacy classes of discrete subgroups of $\mathsf{E_n}$, $\mathsf{O_{n+1}}$ and $grpO^1(1,n)$ that have no elements with fix points.*

4.2 The 2-dimensional case

We will now apply the theory developed above to the most simple situation: $n = 2$.

Theorem 4.5. *There are two complete manifolds with constant curvature $K = 1$: The 2-sphere and the projective plane. Their fundamental groups are the trivial group and $\mathbb{Z}_2$.*

Proof. We must find all subgroups of $\mathsf{S}_2 = \mathsf{O}_3$ that act discretely and without stabilizers on $\mathbb{S}_2$. First of all note that such subgroups are finite because $\mathbb{S}_2$ is compact (and an infinite number of points in a compact set is never a discrete subset). Secondly as every element of O_3 can be diagonalized with eigenvalues 1 or -1 we must conclude that if an element does not fix any point of $\mathbb{S}_2$ it must have only eigenvalues -1 and hence it is the reflection around the zero. This means that we have two possible subgroups: the trivial one and the one generated by the point-reflection. This last group identifies opposite points of the sphere so the quotient is the projective plane. □

Theorem 4.6. *There are 5 types of complete manifolds with constant curvature $K = 0$*

- *the Euclidian plane with trivial fundamental group,*
- *the cilinder with radius r and fundamental group $\mathbb{Z}$,*
- *the Moebius strip with radius r and fundamental group $\mathbb{Z}$,*

- *the torus with fundamental group* $\mathbb{Z} \times \mathbb{Z}$*,*
- *the Kleinian bottle fundamental group* $\langle g, t | gtg^{-1} = t^{-1} \rangle$*.*

Proof. There are two types of elements in E_2 that do not fix any point of the plane: The unsolvability of $Ax + b = x$ implies that $A - I$ is not invertible and b is not an eigenvector of $(A - 1)$ with non-zero eigenvalue. So we have the translations $t_b : x \mapsto x + b$ and glide reflexions $g_{L,\ell}$ which are compositions of a reflection around an axis L and a translation parallel to that axis over a distance ℓ.

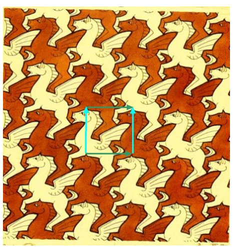
First we classify the groups containing only translations. If the group is generated by one translation we can up to conjugation choose this translation along the X-axis. The quotient is then a cylinder with circumference the length of the translation.

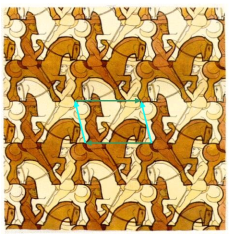
If the group is generated by two translations these two translations cannot be in the same direction. If this were the case either the ratio of there lengths would be rational (and then we could construct the greatest common divisor which generates the two) or irrational (and then we could construct smaller and smaller translations which contradicts the discreteness).

The group cannot be generated by three or more translations because then we could express these extra translations in terms of the first two and if their coordinates were all rational we could again find smaller generators and if there were irrational coordinates we could construct smaller and smaller translations.

The quotient manifold by a group generated by two translations is a torus.

Now consider the general case. If a group contains two glide reflections in different directions then it also contains a rotation so there can only be one direction around which there are glide reflections. The smallest glide reflection must generate all translations in this direction otherwise we can construct smaller glides by composing glides and translations to obtain smaller glide reflections. So there are two extra possibilities for the group: either it is generated by one glide reflection and then the quotient is a Moebius strip or it is generated by a glide reflection and a translation in another direction and then the quotient is a Kleinian bottle.





□

The hyperbolic case is the most difficult. We will not treat it completely

Lemma 4.7. $\mathsf{H}_2 \cong \{A \in \mathsf{GL}_2 | \det A = \pm 1\}/\{1, -1\}$.

Proof. We consider another model of the hyperbolic plane: the upper half plane

$$\tilde{\mathbb{H}} = \{x + iy \in \mathbb{C} | y > 0\}$$

with the metric

$$g_{ij} = y^{-2}\delta_{ij}$$

one can check easily that this also has constant curvature -1. In this model the geodesics are given by vertical lines and half-circles with center on the real axis. For every couple of points there is a unique geodesic line that connects them. Two geodesics intersect in at most one point.

The symmetry group acts of course transitively on the geodesics. We will now obtain a nice description of this group.

We let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{SL}_2(\mathbb{R})$ act on $\tilde{\mathbb{H}}$ as follows

$$z \in \tilde{\mathbb{H}} \mapsto w = \frac{az + b}{cz + d} \text{ if } \det A = 1, \text{ or } w = \frac{a\bar{z} + b}{c\bar{z} + d} \text{ if } \det A = -1$$

This map is well defined because one can check that $\Im w > 0$. It is also not difficult to compute that these maps are isometries. Note that $A = -1$ acts as the identity so we should divide this out.

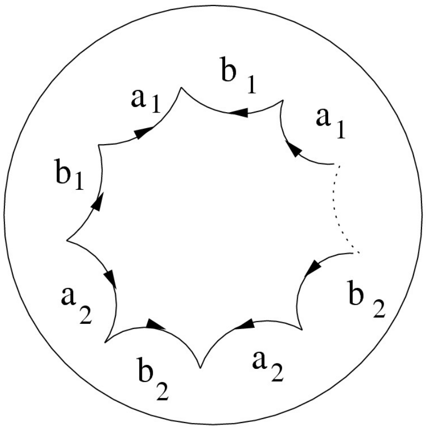
We can see that this group works transitive on the points because we can map i to all points of the upper plane. We can also see that the group that stabilizes i is $O(2)$ so the group above is the group of all transformations. □

In this group we must find discrete subgroups with elements that do not fix points in $\tilde{H}$. We end with an example of such a group.

If we have two geodesic segments $[a_1 b_1]$ and $[a_2 b_2]$ of the same length there is a unique element in PSL_2 that maps a_1 to a_2 and b_1 to b_2. Now construct a $4m$-gon in the following way: Take 2 points $p, q \in \tilde{\mathbb{H}}$ and let ρ be the unique isometry that fixes p and for which $d\rho_p$ is a rotation around $2\pi/4n$. The images of q under ρ^i are denoted by q_i.

Now define the isometries:

$$\alpha_i : a_i = [q_{4i} q_{4i+1}] \to a_i^{-1} = [q_{4i+3} q_{4i+2}]$$
$$\beta_i : b_i = [q_{4i+1} q_{4i+2}] \to b_i^{-1} = [q_{4i+4} q_{4i+3}]$$



One can prove that these isometries generate a discrete subgroup of PSL_2 of which the elements do not fix points of $\tilde{\mathbb{H}}$:

$$\Gamma := \langle \alpha_1, \beta_1 \ldots, \alpha_m, \beta_m | \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots \alpha_m \beta_m \alpha_m^{-1} \beta_m^{-1} = 1 \rangle$$

The quotient of such a group is a surface with m handles. These are not the only discrete subgroups of PSL_2. Such groups are also called Fuchsian groups; a complete study of them is very involved and connects many different areas of mathematics.

Chapter 5

Integrals and differential operators

5.1 The external derivative and integrals

As we already saw a k-form is a antisymmetric $(0, k)$-tensor on M. The space of k-forms is denoted by $\Omega_k(M)$. Contrarily to vector fields we have a canonical notion of deriving a k-form. This derivation is called the external derivative and it transforms a k-form in a $k+1$-form. First we define it on 0-forms which are in fact smooth functions.

$$(df)(X) := X(f) = X^i \partial_i f \text{ or } df := \partial_i f dx^i$$

We extend this operation to k-forms

$$(d\omega)(X_1, \ldots, X_{k+1}) = \sum_i (-1)^i (d\omega(X_1 \ldots X_{i-1}, X_{i+1}, \ldots X_{k+1}))(X_i)$$

It can easily be checked that this is indeed an antisymmetric tensor. In terms of the coefficients we get

$$(d\omega)_{j_1 \ldots j_{k+1}} = \sum (-1)^i \partial_{j_i} \omega_{j_1 \ldots j_{i-1} j_{i+1} \ldots j_{k+1}}$$

The main property of this differential operater is that its square is zero:

$$d^2\omega = 0.$$

The differential operator is a generalization of the differential operators in $\mathbb{R}^3$. If we take $M = \mathbb{R}^3$ we can identify the 0-forms with functions in three variables, the 1-forms with functions from $\mathbb{R}^3$ to $\mathbb{R}^3$ (using the basis dx, dy, dz. The 2-forms are also functions from $\mathbb{R}^3$ to $\mathbb{R}^3$ (using the basis $dy\wedge dz, dz\wedge dx, dx\wedge dy$). Finally the 3-forms can be identified with functions from $\mathbb{R}^3 \to \mathbb{R}$ using the basis $dx\wedge dy\wedge dz$. In this way we get the following diagram

$$\begin{array}{ccccccc}
\Omega_0(M) & \xrightarrow{d} & \Omega_1(M) & \xrightarrow{d} & \Omega_2(M) & \xrightarrow{d} & \Omega_3(M) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
C^\infty(\mathbb{R}^3) & \xrightarrow{\mathsf{grad}} & C^\infty(\mathbb{R}^3)^3 & \xrightarrow{\mathsf{curl}} & C^\infty(\mathbb{R}^3)^3 & \xrightarrow{\mathsf{div}} & C^\infty(\mathbb{R}^3)
\end{array}$$

The fact that $d^2 = 0$ now translates into the well known identities $\mathsf{curlgrad} f = 0$ and $\mathsf{divcurl}\vec{v} = 0$.

Another way to see k-forms as a generalization of standard calculus, is in integration. A k-form can be seen as the thing you put behind the integrand. If ω is a k-form on M and S is a k-dimensional submanifold of M we can define the integral

$$\int_S \omega := \int \dots \int_U \omega(\frac{\partial\sigma}{\partial\theta_1}, \dots, \frac{\partial\sigma}{\partial\theta_k}) d\theta_1 \dots d\theta_k$$

In this formula $\sigma : U \subset \mathbb{R}^k \to M$ is a parametrization of the surface. The nice thing is that this formula does not depend on the parametrization as long as the parametrization have the same orientation (i.e. the jacobian between them has positive determinant) so it is intrinsically determined by S, its orientation and ω.

In standard calculus there are a lot of theorems that connect differential operators and integrals. These theorems are all special cases of the theorem of Stokes

Theorem 5.1 (Stokes). *If S is a k-dimensional manifold in M and ω is a $k-1$-form then*

$$\int_S d\omega = \int_{\partial S} \omega.$$

∂S is the boundary of S inside M. The orientation on ∂S is determined by the orientation on S: If σ is a parametrization of S such that $\sigma(\vec{\theta}) \in S \Rightarrow \theta_1 < 0$ and $\sigma(\vec{\theta}) \in \partial S \Rightarrow \theta_1 = 0$ then $\sigma|_{\theta_1=0}$ gives us the orientation on ∂S.

Finally how do we integrate functions over a manifold? As we have seen above, if M is a manifold the correct thing to integrate over M is an n-form. So we should find a way to turn a function in to an n-form. This can be done using the metric.

In the case of n-forms, the space $\wedge^n T_p^* M$ is one-dimensional (a basis is $dx^1 \wedge \cdots \wedge dx^n$) so we can find a vector in every $\wedge^n T_p^* M$ of size 1 according to the metric. An n-form which in every point has size one is called a volume form.

$$\langle dv, dv\rangle_p = 1.$$

If a volume form exists then we call the manifold orientable. If it exists then there are exactly two possible volume forms

$$dv = |g|^{-\frac{1}{2}} dx^1 \wedge \cdots \wedge dx^n \text{ and } -dv$$

Not every manifold is orientable (e.g. the moebius strip) but every manifold has a cover that is orientable.

Now if M has a volume form V we define the integral of the function f over M to be the integral

$$\int_M f dv = \int_M f |g|^{-\frac{1}{2}} dx^1 \wedge \cdots \wedge dx^n$$

5.2 The Gauss-Bonnet theorem

We end this chapter with a very nice theorem that correlates the curvature of a two dimensional manifold with the Euler characteristic.

Recall that if M is a 2-manifold then a (geodesic) triangle is a simply connected closed piece T of M such that the boundary in M consists of 3 geodesic curves that intersect each other only at their end points. These three points are called the vertices and the three geodesic are called the edges.

A (geodesic) triangulation consists of a set of triangles such that two triangles are disjunct or have either one vertex in common or one edge (and two vertices).

The Euler characteristic of a surface M is the number of triangles (F) and the number of vertices (V) minus the number of edges (E):

$$\chi(M) = F - E + V = V - \frac{3}{2}F + F = V - \frac{1}{2}F$$

($E = 3/2F$ because every triangle has 3 edges and an edge is shared by two triangles). The expression only depends on the manifold and not on the triangulation. It is therefore a topological invariant. The surprising thing is that one can also calculate this number by integration the sectional curvature of the surface.

Theorem 5.2 (Gauss-Bonnet). *If M is a 2-dimensional Riemannian manifold and K is its sectional curvature then the integral of the sectional curvature is 2π times the Euler characteristic.*

$$\int_M K dv = 2\pi\chi(M).$$

Proof. We need to prove that for a geodesic triangle T

$$\int_T K ddv = \alpha + \beta + \gamma - \pi$$

where α, β, γ are the angles of the triangle.

If we then sum over all the triangles of the triangulation we get

$$\int_M K ddv = \sum_i \int_{T_i} K ddv = \sum_i \alpha_i + \beta_i + \gamma_i - F\pi.$$

In every vertex of the triangulation the angles of all triangles that meet this vertex sum up to 2π so

$$\int_M K ddv = 2\pi V - \pi F = 2\pi\chi(M).$$

Chose one of the edges of the triangle and let $\gamma : [0, 1] \to M$ be the corresponding geodesic. in every point $\gamma(t)$ we denote by $N_t \in T_{\gamma(t)}M$ the vector with norm 1 perpendicular to $\gamma'(t)$ and pointing into the triangle. Note that N_t is the parallel transport of N_0 along γ.

We now define a coordinate system by

$$(u, v) \mapsto \exp_{\gamma(u)} vN_t$$

One can check that $\|\partial_v\| = 1$ in every point and

$$\begin{aligned}
\partial_v\langle\partial_u, \partial_v\rangle &= \langle\nabla_{\partial_v}\partial_u, \partial_v\rangle + \langle\partial_u, \nabla_{\partial_v}\partial_v\rangle \\
&= \langle\nabla_{\partial_v}\partial_u, \partial_v\rangle + 0 \\
&= \langle\nabla_{\partial_u}\partial_v, \partial_v\rangle = \frac{1}{2}\partial_u\|\partial_v\| = 0.
\end{aligned}$$

However $f = \|\partial_u\|$ is not always equal to 1. In this coordinate system the metric has a simple form: $g_{uu} = f^2, g_{uv} = 0, g_{vv} = 1$. The sectional curvature now becomes

$$K = -\frac{\partial_v^2 f}{f}.$$

while the volume form is $f du \wedge dv$. The integral can be expressed as

$$-\int_T \partial_v^2 f du \wedge dv = \int_T d(\partial_v f du) = \int_{\partial T} \partial_v f du.$$

Now if $\sigma : [0, 1] \to M$ is a geodesic and $\theta(t)$ denotes the angle between ∂_v and $\sigma'(t)$:

$$\frac{\sigma'(t)}{\|\sigma'\|} = \cos\theta(t)\partial_v - \sin\theta(t)\frac{1}{f}\partial_u$$

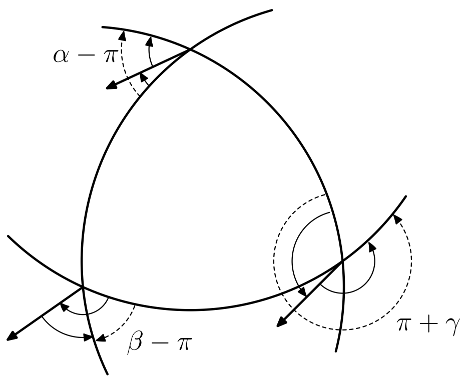
now

$$\begin{aligned}
\theta'(t) &= \frac{-1}{\sin\theta}(\cos\theta)' = \frac{-1}{\sin\theta}\frac{d}{dt}\langle\partial_v, \sigma'(t)\rangle \\
&= \frac{-1}{\sin\theta}(\langle\nabla_{\sigma'(t)}\partial_v, \sigma'(t)\rangle + \langle\partial_v, \nabla_{\sigma'(t)}\sigma'(t)\rangle \\
&= \frac{-1}{\sin\theta}\langle\nabla_{\cos\theta(t)\partial_v - \sin\theta(t)\frac{1}{f}\partial_u}\partial_v, \sigma'(t)\rangle \\
&= \frac{1}{f}\langle\nabla_{\partial_u}\partial_v, \sigma'(t)\rangle = \frac{1}{f}\langle\partial_v f\partial_u, \sigma'(t)\rangle \\
&= -\sin\theta(t)\partial_v f = \partial_v f du(\sigma'(t))
\end{aligned}$$

This means that the integral can be rewritten as

$$\begin{aligned}
\int_{\partial T}\partial_v f du = \int_{\partial T} d\theta &= \int_{E_1} d\theta + \alpha + \int_{E_2} d\theta + \beta + \int_{E_3} d\theta + \gamma \\
&= \theta_{1e} - \theta_{1s} + \theta_{2e} - \theta_{2s} + \theta_{3e} - \theta_{3s} \\
&= \alpha - \pi + \beta - \pi + \gamma + \pi = \alpha + \beta + \gamma - \pi
\end{aligned}$$

Where θ_{is} and θ_{ie} denote the angles between ∂_v and the i^{th} edge at the starting and ending position. This means that $-\theta_{i+1s} + \theta_{ie}$ is the angle at the where E_i ends.



□

Chapter 6

The Ricci Flow

6.1 The Laplacian and the heat equation

Note that in standard calculus $\mathsf{divgrad} f$ is called the Laplacian of f but in the general setting of differential forms this equation does not make sense. What we can do is to take the covariant derivative of the the one form df to obtain a $(0,2)$-tensor $(X,Y) \mapsto \nabla_Y(df)(X)$. To turn this into a function we can raise one of the indices and contract:

$$\Delta f := g^{ij}(\partial_i \partial_j + \Gamma_{ij}^k \partial_k) f.$$

This differential operator is called the Laplacian and it depends crucially on the metric.

In order to make sense of it we need an identification of the 1-forms and the 2-forms, this can only be done using a metric.

A function or a form that satisfies the equation $\Delta f = 0$ is called a harmonic function or form. On a compact closed manifold the only harmonic functions are the constant functions. This can easily be seen because if M is compact and f non-constant then there is a point $p \in M$ for which f is maximal and strictly maximal in at least on direction. Choose normal coordinates around this point then we get that $\Delta f(p) = \partial_i \partial_i f$. Because f is maximal all second derivatives are negative and at least in one direction strictly negative so $\Delta f(p) < 0$.

Note that on a closed manifold the integral

$$\begin{aligned}\int_M \Delta f dv &= \int_M (g^{ij}(\partial_i\partial_j + \Gamma^k_{ij}\partial_k)f)\frac{1}{\sqrt{|g|}}dx^1\dots dx^n \\ &= \int_M \partial_i(\sqrt{|g|}g^{ij}\partial_j f)dx^1\dots dx^n \\ &= \int_M d((-1)^{j-1}\sqrt{|g|}g^{ij}\partial_j f dx^1\dots d\hat{x}^j\dots dx^n) = 0\end{aligned}$$

The last equality follows from the theorem of Stokes and the fact that ∂M is empty.

Vice versa we have the theorem

Theorem 6.1. *If $\int_M hdv = 0$ then the differential equation*

$$\Delta f = h$$

has a unique solution up to a constant.

Proof. One can show the existence of a function

$$G : M \times M \to \mathbb{R}$$

such that for a smooth h the function

$$f(y) = \int_{M\times\{y\}} G(x,y)h(x)dv$$

satisfies

$$\Delta f = h - \frac{\int_M hdv}{\int_M dv}.$$

The function G is called Green's function and in dimension 2 it has the form

$$G(x,y) = -\frac{1}{2\pi}\ln\sin\frac{d(x,y)}{2}$$

where $d(x,y)$ is the distance between x and y coming from the metric. □

The Laplacian is an operator that is important in many physical problems. One of them is the heat equation

Miniature 5: The Heat Equation

Consider an iron ring of radius 1. The temperature on the rod can then be represented by a function $u : [0,2\pi]\times\mathbb{R}\to\mathbb{R}$. The first coordinate ($\theta$) represents the place on the rod, the second represents the time (t). To calculate the evolution of the temperature distribution physicist use the heat equation:

$$\partial_t u = k\partial_\theta^2 u = k\Delta f$$

where k is a constant. This equation has solutions of the form

$$ae^{-b^2kt}\cos(b\theta + c).$$

So if we know $u(\theta, 0) = f(\theta)$ we can use the Fourier series of f

$$a_0 + a_1\cos(1\theta + c_1) + a_1\cos(2\theta + c_2) + \dots$$

to solve the general equation. For all but the constant the amplitudes of the cosine functions decrease exponentially so we have that $\lim_{t\to\infty} u(x,t) = a_0 = (2\pi)^{-1}\int_0^{2\pi} u(x,0)dx$. The equation smoothens the function until it becomes constant.

To finish the proof of the Poincaré conjecture in dimension 2 we need to prove that every compact manifold admits a metric of constant curvature. The technique to do this could be to find a differential equation for the metric such that the curvature of the metric satisfies an equation similar to the heat equation. This will be possible using the Ricci Flow.

6.2 The Ricci Flow

The metric that is constructed this way does not necessarily have constant curvature. The thing we want to do is to adjust this metric by smoothening out the curvature. This is done using the ricci flow equation which is an adaptation of the heat equation.

We let $g : \mathbb{R} \to \mathsf{Ten}^{0,2}(M)$ be a smooth family of metrics on M. We say that this family satisfies the normalized Ricci flow if

$$\partial_t g_{ij} = -2\mathrm{Ric}_{ij} + \frac{2}{n}Rg_{ij}$$

where

$$\bar{R} = \frac{\int_M R dv}{\int_M dv}$$

is the average (mean) of the scalar curvature and n is the dimension of the manifold. This normalized equation preserves the volume of the metric $\int_M d\mathrm{vol}$.

The factor of -2 is of little significance, since it can be changed to any nonzero real number by rescaling t. However the minus sign ensures that the Ricci flow is well defined for sufficiently small positive times; if the sign is changed then the Ricci flow would usually only be defined for small negative times. (This is similar to the way in which the heat equation can be run forwards in time, but not usually backwards in time.)

Miniature 6: Richard Hamilton (1943 -)

Richard Streit Hamilton is Professor of Mathematics at Columbia University. He received his Ph.D. in 1966 from Princeton University.
Hamilton is best known for having invented the Ricci flow, which Grigori Perelman employed in his proof of the Thurston geometrization conjecture and the Poincar conjecture.
Hamilton was awarded the Oswald Veblen prize in 1996 and the Clay Research Award in 2003.

What one wants to do now is to find a solution for the Ricci flow starting from a general metric define on M. The hope is then that this flow will evolve to a metric of constant curvature. In dimension 2 this hope is indeed justified as we will prove in this chapter. In dimension 3 however things are more subtle. Firstly the equation might develop singularities: places where the metric becomes infinite or degenerate and secondly the result of the flow does not always give a metric of constant curvature. The second problem comes from the fact that instead of the 3 model geometries in 2-dimensions (the sphere, the euclidean and the hyperbolic plane) there are 8 model geometries in 3 dimensions (this is called Thurston's geometrization conjecture). The first problem is solved using surgery: cutting out the singularities.

Let us restrict to the 2-dimensional case

Lemma 6.2. *If* $n = 2$ *then* $2\mathrm{Ric} = Rg$

Proof. Because of the symmetry the only nonzero components of R_{ijkl} are

$$R_{1212} = -R_{2112} = -R_{1221} = R_{2121}.$$

This means that $\mathrm{Ric}_{ij} = g^{kl}R_{kilj} = g^{i'j'}R_{1212}(-1)^{i+j}$ where $i' = nei$ and $j' \neq j$. Therefore $(-1)^{i+j}g^{i'j'}$ is the i, j-minor of g^{ij} and hence

$$\mathrm{Ric}_{ij} = |g|^{-1}g_{ij}R_{1212}.$$

and $R = g^{ij}|g|^{-1}g_{ij}R_{1212} = 2|g|^{-1}R_{1212}$ so $Rg_{ij} = 2\mathrm{Ric}_{ij}$. □

Because of this lemma the normalized equation simplifies to

$$\frac{\partial g_{ij}}{\partial t} = (\bar{R} - R)g_{ij}.$$

This shows us that g is a constant solution if and only if its curvature is constant.

Lemma 6.3. *The normalized Ricci flow preserves the volume of the metric and the average curvature.*

Proof. We have that $\mathrm{vol} = \int_M \sqrt{|g|}dxdy$ so $\partial_t \mathrm{vol} = \int_M \sqrt{\det(\bar{R} - R)g}dxdy = \bar{R}\mathrm{vol} - \int_M R\sqrt{|g|} = \bar{R}\mathrm{vol} - \bar{R}\mathrm{vol} = 0$.

Secondly we know from the theorem of Gauss-Bonnet that $\int_M R\sqrt{\det g}$ does not depend on the metric on M at all because it is a topological invariant. The quotient of this integral with the volume is the average curvature and is hence constant. □

Now we want to transform the equation for the metric to an equation for the curvature. To do this we need some calculations

Lemma 6.4. *The expressions for the time derivatives of the curvature and Christoffel symbols are*

$$\begin{aligned}
\partial_t \Gamma^k_{ij} &= \frac{1}{2}(\nabla_i \partial_t g_{jk} + \nabla_j \partial_t g_{ik} - \nabla_k \partial_t g_{ij})g^{kl} \\
\partial_t R^i_{jkl} &= \nabla_j \partial_t \Gamma^i_{kl} - \nabla_k \partial_t \Gamma^i_{jl} \\
\partial_t R_{ij} &= \nabla_i \partial_t \Gamma^k_{kj} - \nabla_k \partial_t \Gamma^k_{ij} \\
\partial_t R &= R_{ij}\partial_t g^{ij} + g^{kl}\nabla_k \nabla_l (g^{ij}\partial_t g_{ij}) - g^{ik}\nabla_k g^{jl}\nabla_l \partial_t g_{ij}.
\end{aligned}$$

Proof. All the expressions above are tensors so we can check them in any coordinate system we want. For the sake of simplicity we use a normal coordinate system around p. In this coordinate system we have that in p $g^{ij} = g_{ij} = \delta_{ij}$ and $\nabla_i T_{kl} = \partial_i T_{kl}$. Also keep in mind that $\partial_t g^{ij} = -g^{im}g^{jn}\partial_t g_{mn}$. So

$$\begin{aligned}
\partial_t \Gamma^k_{ij} &= \partial_t \frac{1}{2}(\nabla_i \partial_t g_{jk} + \nabla_j \partial_t g_{ik} - \nabla_k \partial_t g_{ij})g^{kl} \\
&= \frac{1}{2}(\nabla_i \partial_t g_{jk} + \nabla_j \partial_t g_{ik} - \nabla_k \partial_t g_{ij})g^{kl} + \Gamma^m_{ij}g^{nk}g_{mn} \\
&= \frac{1}{2}(\nabla_i \partial_t g_{jk} + \nabla_j \partial_t g_{ik} - \nabla_k \partial_t g_{ij})g^{kl}
\end{aligned}$$

The rest of the calculations are similar. □

We use the last formula to obtain a differential equation for the curvature:

$$\begin{aligned}
\partial_t R &= \frac{1}{2}Rg_{ij}g^{ik}g^{jl}(-\bar{R} + R)g_{kl} + g^{kl}\nabla_k \nabla_l (g^{ij}(\bar{R} - R)g_{ij}) - g^{ik}\nabla_k g^{jl}\nabla_l (\bar{R} - R)g_{ij} \\
&= R(R - \bar{R}) + 2\Delta(\bar{R} - R) - \Delta(\bar{R} - R) \\
&= R(R - \bar{R}) + \Delta(\bar{R} - R)
\end{aligned}$$

So if we define curvature difference as $\rho = R - \bar{R}$, it satisfies the following differential equation

$$\frac{\partial}{\partial t}\rho = \Delta\rho + \rho^2 + \bar{R}\rho.$$

The final technique we need to tackle the problem is the maximum principle.

Lemma 6.5 (The maximum principle)**.** *If M is a manifold equipped with a continuous family of metrics g_t and $u_t : M \to \mathbb{R}$ is a family of functions that satisfies the inequality*

$$\frac{\partial}{\partial t}u \geq \Delta_{g_t}u + Cu.$$

If $u_0(x) \geq k$ for all $x \in M$ then $u_t(x) \geq ke^{Ct}$ for all t

Proof. first we substitute $u = e^{Ct}\tilde{u}$ to obtain the inequality

$$\frac{\partial}{\partial t}\tilde{u} \geq \Delta_{C^{-1}g_t}\tilde{u}.$$

So we restrict to the case where $C = 0$. Now define $u^\epsilon = u + \epsilon(1+t)$. If for some $\epsilon > 0$ there is a $x \in M$ with $u^\epsilon_t(x) < c$ then we can also find a time t_1 such that $\exists x_1 \in M : u^\epsilon_{t_1}(x_1) = c$ for the first time.

We note that because u^ϵ is minimal in t, the partial time derivative is negative in x_1. In space coordinates the point is also minimal so the Laplacian is positive, but this contradicts the fact that

$$\frac{\partial}{\partial t}u^\epsilon \geq \nabla_{g_t}u^\epsilon + \epsilon > 0.$$

So $u^\epsilon > c$ for all $\epsilon > 0$ and hence $u \geq c$. □

We are now ready to formulate the main result about the Ricci Flow in two dimensions.

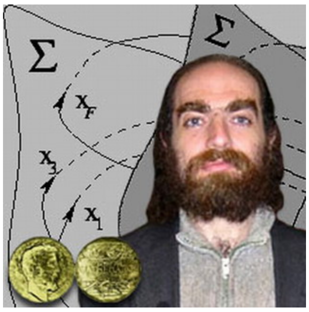
Theorem 6.6. *If (M, g_0) is a 2-dimensional Riemannian manifold then there exist a unique solution g_t to the normalized ricci flow such that $g_\infty := \lim_{t\to\infty} g_t$ is a metric of constant curvature i.e. $\rho_\infty = 0$.*

The proof of this theorem is covered by the appendix.

This concludes the proof of the Poincaré Conjecture in 2 dimensions: We have proved that every compact 2-dimensional Riemannian manifold admits a metric of constant curvature. If this manifold is simply connected then we know that it is $\mathbb{S}_2$, $\mathbb{E}_2$ or $\mathbb{H}_2$. Only the first one of these is compact. The proof in dimension

2 could be simplified in many ways, but the way it has been described in these notes illustrates best the philosophy behind the proof in dimension 3 by Grisha Perelman.

Miniature 7: Grisha Perelman (1966 -)



Grigori Yakovlevich Perelman born 13 June 1966 in Leningrad, is a Russian mathematician who has made landmark contributions to Riemannian geometry and geometric topology. In particular, he has proved Thurston's geometrization conjecture. This solves in the affirmative the famous Poincar conjecture.
In August 2006, Perelman was awarded the Fields Medal. The Fields Medal is widely considered to be the top honor a mathematician can receive. However, he declined to accept the award or appear at the congress.