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Interview Abel Prize laureate 2021 László Lovász

On graphs and graphons

On the 17th of March the Norwegian Academy of Science and Letters awarded the Abel Prize for 2021 to László Lovász and Avi Wigderson for their contributions to discrete mathematics and theoretical computer science. Furthering our newly started tradition of interviewing prize winners, our editors Raf Bocklandt and Nicos Starreveld arranged a zoom meeting with László Lovász to talk about his ideas, proofs, algorithms and conjectures.

At the Gymnasium

László Lovász was born in Hungary in 1948. He grew up in Budapest and already at a young age he excelled in mathematics. This led him to enroll in the Fazekas Mihály Gimnázium, a school with a special program aimed at students interested in mathematics.

"Nobody in the family was a mathematician but when I was in 8th grade, that's the last grade of elementary school in Hungary, I attended a math club. The leader of the club recommended that I go to a certain high school, which had just started a special math class. I was very excited to join this class and it turned out to be an absolutely great four years. Some of my classmates became quite important in mathematics, so it was an excellent group. We were good friends even though we were rivals at various mathematics competitions, such as the International Mathematical Olympiad and a math quiz on tv. This class not only got me hooked on mathematics for my whole life, but I also met my wife Katalin Vesztergombi there, she was a classmate of mine and a very good mathematician too.

The school had a semi-formal connection with the university, so some people from there — mainly young professors but also some senior ones — came to the school and gave lectures. Because the class was a novelty, people were very interested in meeting this group of students. Probably the most important among these people was Paul Erdős, who came and gave a couple of talks. He could give very elementary talks about a number of fascinating unsolved problems. Probably one of his

A young Lovász working in prime-time on a math problem

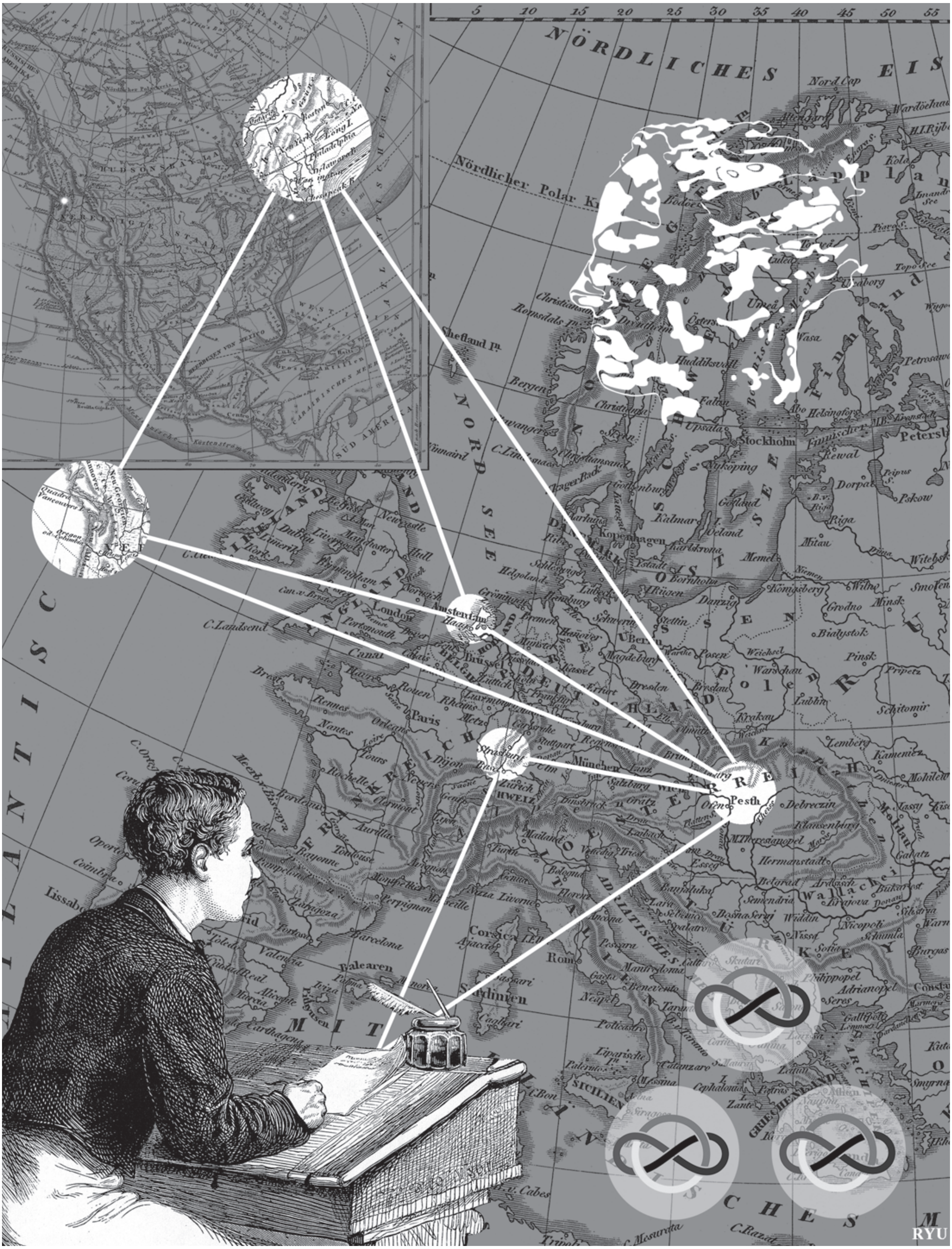

Illustration: Ryu Tajiri

most important contributions was to recognize and advertise that elementary problems can be difficult and important, even though a high school student can understand what the problem is.

A few years earlier Erdős had already met one of my classmates, Lajos Pósa. They worked and they published some joint papers, which are important papers on graph theory and are still being cited. I got interested because this classmate challenged me to find a proof for a joint result he had with Erdős. In a day or two I could prove it, so I was introduced to Erdős and he listened to my solution. It was a very characteristic thing for him that he put a footnote in the paper saying that Lovász has independently proven this, although it's not really independent. I mean, if you know that something is true, it's much easier to prove. He always found it very important to make young people recognized and to help them along their career. Lajos Pósa became interested in math education and he is a very important person even until today."

Kneser's conjecture

One of the things Lovász is famous for, is his proof of Kneser's conjecture [8]. Formulated by Martin Kneser in 1955, it states that the graph whose vertices correspond to the k-subsets of a set with n elements and whose edges connect disjoint subsets has chromatic number $n-2k+2$, if $n \geq 2k$. In 1978 Lovász proved this conjecture using a famous result from topology: the Borsuk–Ulam theorem. This tells us that for every continuous map from the n-sphere to $\mathbb{R}^n$ there are two opposite points on the sphere with the same image.

"Using topology in combinatorics also had to do with Erdős. He wrote a number of papers with András Hajnal, a very distinguished combinatorialist at that time. In particular they were interested in graphs which have a high chromatic number and do not contain short cycles, or an easier problem: they do not contain short odd cycles. Kneser graphs share this property together with Borsuk graphs. The latter are obtained by taking a sufficiently dense finite set of points on the sphere and connecting two of them if they are almost opposite. If you adjust the parameters, this graph has a high chromatic number due to Borsuk's theorem. On the other hand, it doesn't have short odd cycles because you have to move two steps to be close to a point again. So you had two constructions and one of them was only conjectured. The natural idea was to look at Borsuk's theorem and see whether it can be applied to the other case. I wrote my master's thesis in topology so I knew a fair bit of topology, not really the most cutting edge theories but sort of the basics like homology theory. It took a year or two to really work it out but in the end I succeeded."

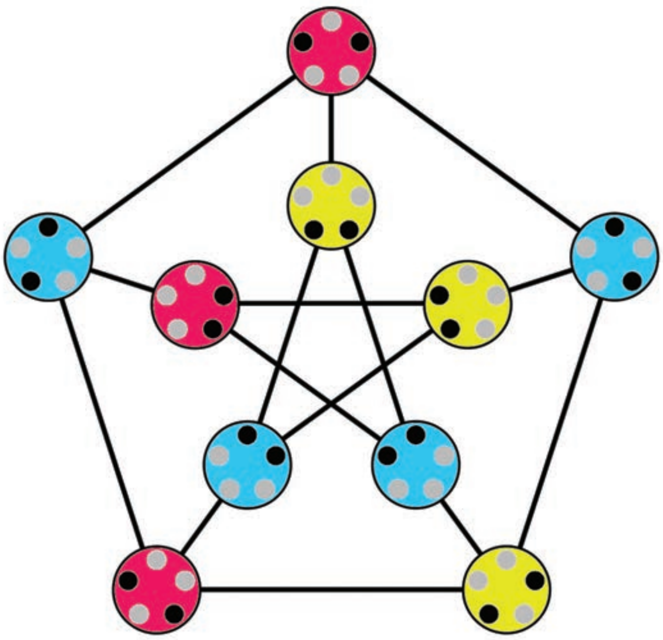

Figure 1 The Kneser graph with $(n,k)=(5,2)$ has chromatic number $n-2k+2=3$ [17].

The Dutch connection

Lovász has a strong connection with the Netherlands. He received the Brouwer Medal in 1993 and since 2006 he is a foreign member of the Royal Netherlands Academy of Arts and Sciences (KNAW). These close ties can be traced back to the late seventies and early eighties when he started collaborating with Lex Schrijver and Hendrik and Arjen Lenstra.

"When I was a professor at the university of Szeged, Lex Schrijver came to visit me and we shared an office for a year. We had a lot of common ideas which eventually resulted in many joint papers and a joint book. It's a very close research connection and a very close friendship.

There was one theorem we worked on, which had a sort of side condition which we didn't like. So the question was whether you can get rid of this side condition. I realized that this could work if we could do the classical Dirichlet's approximation theorem algorithmically. I mean not only show the existence of rational approximations of a real number with the same denominator but also provide an efficient construction. That's not an easy thing because the proof usually uses the pigeonhole principle and you really have to go through the whole list of elements until you find it."

Dirichlet's approximation theorem states that you can always approximate a real number r within a margin smaller than $1/qN$ by a fraction p/q with denominator $q \leq N$. This approximation can also be extended to higher dimensions and this turned out to be helpful in the study of lattices.

"I met Hendrik Lenstra in Bonn at a conference where he gave a talk about lattice reduction. The method he presented was a non-polynomial time method and it only worked in fixed dimension. I thought then that maybe this lattice reduction could be extended to higher dimensions so that its complexity only grows polynomially within a specific dimension. I began to experiment with different algorithms and eventually I managed to find an algorithm of which I could prove that it is polynomial.

I remember I was experimenting on a programmable calculator, I think it was a Texas Instruments 59, which calculated to thirteen decimal places, so you could get very accurate calculations. It was good for

Lex Schrijver, László Lovász and their coauthor Martin Grötschel

showing that certain ideas would not work. Faulty algorithms would quickly grind to a halt after using it two or three times. It was great when I first saw an algorithm that seemed to be working. I wrote to Hendrik that I had this algorithm and he answered that he just found out with his brother that if it worked then they could factor polynomials in polynomial time. This was of course a very nice open problem. So that's how this joint paper arose which we published in *Mathematische Annalen*."

The LLL-algorithm as it is known today is a cornerstone of modern computational mathematics. It has many applications in pure and applied mathematics but also in computer science and cryptography [13].

"We were not aware of the computer science applications at that time so the main application of our result was factoring polynomials. The real application, that came maybe a few years later, was discovered by Lagarias and Odlyzko [7] who observed that they could apply the algorithm to break the so-called low-density knapsack code, so that's how it started to have applications in cryptography."

A new theory of large graphs

In 1999 Lovász moved to Microsoft Research with his wife, mathematician Katalin Vesztergombi, to work with the research group led by Jennifer Chayes.

"Together with Katalin we were invited to visit Microsoft Research, which was a new institution at that time. Jennifer Chayes was the head of the theory group, which consisted of maybe four people at that time. Later it grew to maybe 15 or 20 people. It was a very small but also a really excellent research group. There was no pressure doing anything useful, the only pressure was that they said that you have to keep your door open. Meaning that if somebody comes in, a software engineer for example, you had to listen and discuss if you had any ideas. In fact, quite soon after I arrived, maybe half a year later, somebody came with a really nice question which we managed to solve with some effort. Together with my wife Katalin and Van Vu, a postdoc back then and currently a professor at Yale University, we came up with an algorithm that software engineers were absolutely happy with. This increased the reputation of the group."

During the seven years in Microsoft, these research results were published in a series of articles [1, 2, 3, 11] and summarized in a book written by Lovász [12]. These articles formed the foundations of the beautiful theory of graph-limits. One of the novelties of this theory is that it showed how large graphs could be studied using tools from functional analysis. The driving question was posed by Chayes: "How can you define convergence of a sequence of graphs and what can the limiting object be?" Lovász has been investigating this question for a long time. Some ideas behind the approach of the Microsoft team were born much earlier.

"It really goes back to my high school years, and even though I don't like to brag about this I really think it's nice to share. Graph theory was not really respected in those days and people thought it was just about puzzles. I wanted to make a more serious theory of graphs, in particular to

Peter van Emde Boas, László Lovász, Hendrik Lenstra and Arjen Lenstra in Bonn

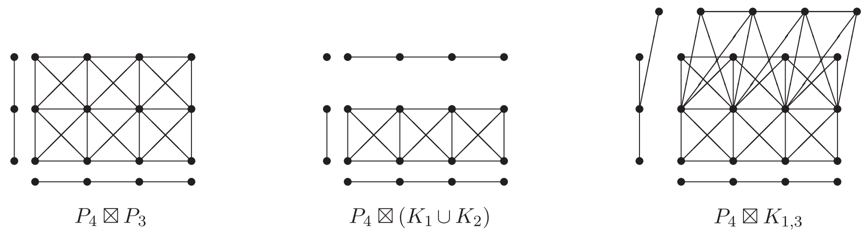

Figure 2 Examples of the strong product [15].

develop an algebraic approach to graph theory. A question that I was intrigued by was whether we can take a product of two graphs for example. Because if you have a product then you want to check whether it is commutative and associative or whether there is a cancellation law: consider graphs G, H and W and suppose that $G \boxtimes H$ is isomorphic to $G' \boxtimes W$ and G is isomorphic to G', does this imply that also H and W are isomorphic?. The product that I found back then, which was already known to mathematicians but I was unaware of this, was the strong product of graphs [9]. With some non-trivial amount of work I could prove a cancellation law for this product [10]. The key idea was to look at a sequence of numbers corresponding to homomorphisms, because that's a multiplicative sequence so you automatically have a cancellation law."

A graph $G = (V, E)$ can be seen as a set with a reflexive and symmetric relation by putting uRv if u and v are equal or joined by an edge. A homomorphism between two graphs is defined as a morphism between the relations (a map $\phi : V(G) \to V(H)$ such that $uRv \Rightarrow \phi(u) R \phi(v)$). In this setting the strong product of graphs $G \boxtimes H$ corresponds to the direct product of these relations ($(u_1,u_2)R(v_1,v_2) \Leftrightarrow u_1 R_1 v_1$ and $u_2 R_2 v_2$) and therefore a morphism $\phi : F \to G \boxtimes H$ is determined by its projections onto G and H. This implies that

$$\begin{aligned}&\#\mathrm{Hom}(F, G \boxtimes H)\\ &\quad = \#\mathrm{Hom}(F,G) \cdot \#\mathrm{Hom}(F,H).\end{aligned}$$

In [10] Lovász showed that the sequence of numbers $\#\mathrm{Hom}(-,G)$ determines G up to isomorphism. Furthermore they are all nonzero because you can map any graph to just one vertex of G. Therefore

$$\begin{aligned}&A \boxtimes B \cong A \boxtimes C\\ &\quad \Rightarrow \#\mathrm{Hom}(-, A \boxtimes B) = \#\mathrm{Hom}(-, A \boxtimes C)\\ &\quad \Rightarrow \#\mathrm{Hom}(-,A)\,\#\mathrm{Hom}(-,B)\\ &\qquad\quad = \#\mathrm{Hom}(-,A)\,\#\mathrm{Hom}(-,C)\\ &\quad \Rightarrow \#\mathrm{Hom}(-,B) = \#\mathrm{Hom}(-,C)\\ &\quad \Rightarrow B \cong C.\end{aligned}$$

These homomorphism numbers can also be used to count more intricate things such as the number of embeddings of one graph into another.

"In the seventies we were investigating with Paul Erdős and Joel Spencer the possible numbers of triangles, numbers of pentagons, and so on; we wanted to understand how these numbers relate to each other in very large graphs. At that time we could only prove some starting results.

Around 2002 when I was at Microsoft Research, Michael Freedman was in the group and he was working on quantum computing. He was trying to use topological methods to improve the stability of quantum computation. He was interested in a statistical physical model with a specified partition function. Freedman's question could be translated to a problem of characterizing the sequence of homomorphism numbers of different graphs into a fixed graph. Together with Freedman and Lex Schrijver we solved this problem by formulating a semi-definiteness condition on these homomorphism numbers."

The concept of homomorphism density

$$t(G,H) = \#\mathrm{Hom}(G,H) \,/\, \#V(H)^{\#V(G)}$$

which measures the probability that a random map $\phi : V(G) \to V(H)$ is a homomorphism, combined with a suitable metric on graphs called the cut-distance, are the building blocks of the theory of graph-limits. During his time at Microsoft, Lovász managed to prove, together with Christian Borgs, Jennifer Chayes, Balász Szegedy, Vera Sós and Katalin, that if a sequence of dense graphs $(G_n)_{n \geq 0}$, where the degree of each vertex scales as the size of the graph, has the property that for every fixed graph F, the homomorphism density $t(F, G_n)$ converges to a limit, then there is a natural 'limit object', namely a symmetric measurable function $W : [0,1]^2 \to [0,1]$ called a graphon [12].

"It was again a sort of a lucky coincidence, we were looking, maybe just sort of on a chatting level, at Albert–Barabási graphs. In Albert–Barabási graphs you keep adding vertices, so it's a growing graph sequence. Then we began to check in how many other ways you can let a graph grow. Jennifer asked if there is a kind of limit distribution of these graphs just like there is a limit distribution for numerical distributions."

By subdividing the interval $[0,1]$ in $\#V(G)$ equal parts, you can represent every graph G as a symmetric function W_G defined on the unit square taking values 0 or 1 depending on whether the corresponding nodes are connected by an edge. All homomorphism densities $t(F,G)$ can be expressed as certain integrals in terms of the function W_G. The graphon associated to a sequence of dense graphs is a function $W : [0,1]^2 \to [0,1]$ that can be used to determine the limits of the homomorphism densities in the same way.

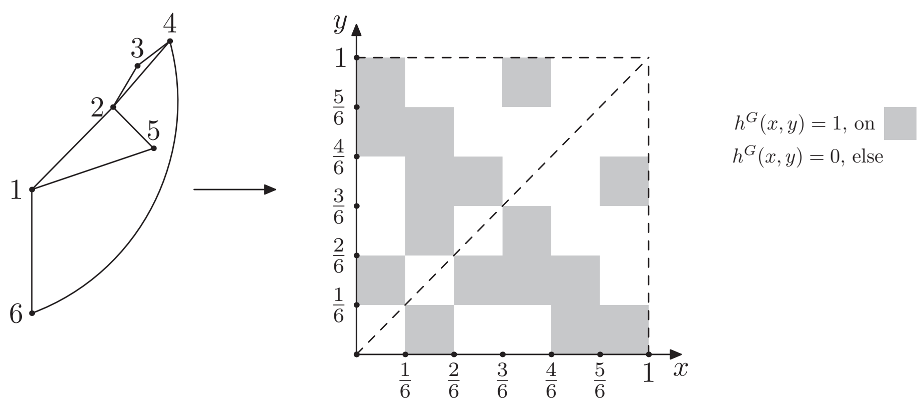

Figure 3 Turning a graph into a graphon [5].

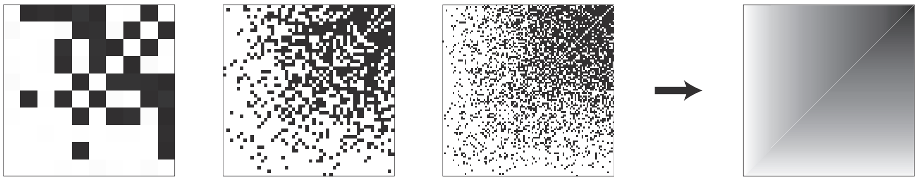

Figure 4 A graphon as a limit of graphs [4].

Math and politics

Apart from the fourteen years he spent in Yale and at Microsoft, Lovász spent most of his professional life in Hungary. During his career the political climate has changed quite drastically.

"In the communist era university life was mostly okay in mathematics. That was of course different in social sciences, where the ideology had to be observed. People who were religious were in principle not allowed to teach, but they could go to the research institutes. Of course, if they were sufficiently distinguished people then nobody really raised a question. It was well-known, for example, that there were two distinguished professors of mathematics at the university that were religious and went to the church, but nobody really wanted to make a note of this.

In mathematics the restrictions that mattered most were those for going abroad, which got more relaxed over the years. Actually, in a sense I should say these were not so bad, because it meant that many excellent mathematicians who lived in Budapest and other cities, did not travel much. Only once every two years they were allowed to go to a conference, so they had ample time to work with students and we got a lot of attention. I learned a lot from these people.

In 1968 I was invited to go to Oberwolfach as a young student. I had a lot of trouble and eventually I only arrived for part of the meeting, but nevertheless it was still very interesting and stimulating to meet people like Tarski. There was always a lot of paperwork in connection with every trip, but I almost never had trouble with traveling to conferences. You had to know that there is a limit: you don't want to ask to go to five conferences a year because that would have been impossible."

A couple of years after the fall of the iron curtain Lovász moved to the United States. He returned to Budapest in 2006 where he became the director of the Mathematical Institute at the Eötvös Loránd University and a professor in computer science. Until 2020 he was also the President of the Hungarian Academy of Sciences. In this role he had some first-hand experience with the new government that has come into power in the last decade.

"It's unclear what its effect on universities is. The universities are in bad shape because of all sorts of reasons such as underfinancing and too much centralization. Financially the universities don't get enough support to actually do their job properly. Whether this will change or not, I don't know. I was the president of the Academy until a year ago and in the last two years the government decided to take away the research institutes from the Academy. Of course there are many models for research in the world. There are some countries without research institutes, while others do have them. Sometimes research belongs to the Academy, sometimes it does not.

Usually the price you pay for a drastic reorganization is too high. There are always things that were not considered in advance and then people lose. For two years the energy of everyone was focused on this reorganization. I think it was a bad idea also for us, it was a really unpleasant experience with the current government. I met Orbán once at some kind of award ceremony and afterwards we talked a little bit at the dinner party. He can be actually a charming person but I wish he had more patience to actually listen to other people's points and try to balance that. I know that's not easy because even within the Academy it was often difficult to balance different interests and different cultures."

Coming up with conjectures

A couple of years ago Lovász started a cooperation with Albert-László Barabási who is very famous in network theory. This resulted in new research that actually became very relevant in light of the recent pandemic.

"Network science is becoming an important paradigm in all sorts of sciences: brain science, social sciences, et cetera, and the idea was that in essence graph theory and network theory are just two names of the same thing, so maybe we should find common grounds. One thing one has to realize is that the two areas have very different styles. A mathematician will not have an idea and in two weeks time write a paper about this, but in other areas that's the standard. We both had to accept that and work out some way of doing research together.

Then the epidemic came and since the topic of the project is the dynamics of large networks, the typical example of such dynamics is the spreading of a disease over a social network. I started research with a small mixed group of network scientists, physicists, computer scientists and mathematicians and that has been very useful. One thing I realized is that you have to do simulations. Without that you don't develop a feeling of what the questions are or what you can expect. To do a good simulation is a non-trivial question. For example you don't want to simulate on random graphs, you have to simulate things on graphs which have some resemblance to real life networks. To set up such a graph is again a non-trivial question. You can have simulations on all sorts of graphs, but from a mathematical point of view you really want something which works for all graphs, or all graphs which have desired properties. You want a general mathematical theorem and ideally also an algorithm.

There is a chain of events: you run simulations, you observe something strange or maybe unexpected and then you try to find a mathematical reason for that. It is quite interesting and I hope that eventually we can prove theorems that will be inspired by these observations.

Often it's not easy, to come up with the right question. It took half a year until we began to observe something, that was there from the beginning in the simulations. A simulation produces numbers, or maybe graphs, but to really gain understanding you also need theorems. I am sort of fighting a war on two fronts because I tried to say that simulations and computer experiments are very important if you want to deal with large-scale phenomena, but you cannot give up on obtaining rigorous results.

Maybe it takes a lot of time because you need to know the theory of random graphs and all sorts of other related things. You have to combine the two and that's why I think both of them are justified. How the community will value one or the other is still an unsettled question. So I guess experimenting, looking for patterns and finding the right conjecture is an important part of mathematics."

Finding a good conjecture can come in many ways: through simulation on a computer, experimenting with pen and paper, or just a discussion among mathematicians at a party. A famous example of the latter is the Erdős–Faber–Lovász conjecture. It states that the union of k complete graphs, each with k vertices, for which every pair of complete graphs shares at most one vertex, can be properly colored with k colors.

This seemingly innocuous question posed at a party in 1972 remained open until this year when a solution was announced for large enough k by Dong Yeap Kang, Tom Kelly, Daniela Kühn, Abhishek Methuku and Deryk Osthus.

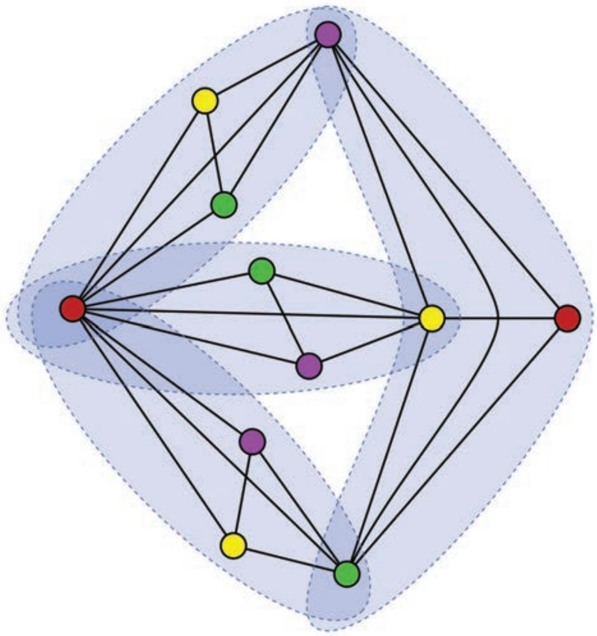

Figure 5 An example of the EFL-conjecture with $k=4$ [16].

"In 1972 there was a month long summer school or seminar at Ohio State University and we were there. It focused on the idea that questions of graph theory often can be generalized to the hypergraphs or graphs where the edges have more than two endpoints. In particular the question is how many colors you need to color the edges of a hypergraph so that no two edges with the same color meet. We discussed this a lot and then somehow three of the participants were meeting a few weeks later in a completely different part of the US. We had this party and then this question came up. Erdős actually didn't believe it, I believed it a little bit more but I wouldn't have been surprised to come up with a trivial counter-example. But then you try this and try that, and you see that it remains true for this special case and that special case. The idea of a counter-example takes you exactly there but not beyond. You never know whether such a question will be easy or difficult.

Recently, Daniela Kühn gave an invited talk at the European Congress and now we know that it's true at least if the parameters are large enough. I always find it curious that you come up with a conjecture based on small values and later somebody finds proof that is valid for very large values. It's almost paradoxical but that's how our current techniques work. Part of the method is probabilistic, which means there are error terms you have to estimate and control. This often only works for large values of the parameters."

Apart from the EFL-conjecture Lovász has three more tantalizing problems on offer: The Lovász conjecture (Every finite connected vertex-transitive graph contains a Hamiltonian path), the Kannan–Lovász–Simonovits conjecture (which recently saw a breakthrough [14]) and the Erdős–Lovász Tihany Conjecture. Enough challenges for motivated readers who want to prove, disprove or just nibble a small, large or even infinite chunk of the parameter space.

References

1 C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós and K. Vesztergombi, Graph limits and parameter testing, in *Proceedings of the Annual ACM Symposium on Theory of Computing*, 2006, pp. 261–270.

2 C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós and K. Vesztergombi, Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing, *Advances in Mathematics* 219(6) (2007), 1801–1851.

3 C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós and K. Vesztergombi, Convergent sequences of dense graphs II. Multiway cuts and statistical physics, *Annals of Mathematics* 176(1), (2012), 151–219.

4 D. Glasscock, What is a graphon?, *Notices of the AMS* 62 (2015), 46–48.

5 F. den Hollander, M. Mandjes, A. Roccaverde and N.J. Starreveld, Ensemble equivalence for dense graphs, *Electronic Journal of Probability* 23 (2018), 1–26.

6 D.Y. Kang, T. Kelly, D. Kühn, A. Methuku and D. Osthus, A proof of the Erdős–Faber–Lovász conjecture (2021), arXiv:2101.04698.

7 J.C. Lagarias and A.M. Odlyzko, Solving low density subset sum problems, *Journal of the ACM* 32(1) (1985), 229–246.

8 M. de Longueville, 25 years proof of the Kneser conjecture: The advent of topological combinatorics, *EMS Newsletter* 53 (2004), 16–19.

9 L. Lovász, Über die starke Multiplikation von geordneten Graphen, *Acta Mathematica Academiae Scientiarum Hungaricae* 18(1–2) (1967), 235–241.

10 L. Lovász, On the cancellation law among finite relational structures, *Periodica Mathematica Hungarica* 1(2) (1971), 145–156.

11 L. Lovász and B. Szegedy, *Limits of Dense Graph Sequences*, Technical Report, Microsoft Research, 2004.

12 L. Lovász, *Large Networks and Graph Limits* American Mathematical Society, 2013.

13 P.Q. Nguyen and B. Vallee, *The LLL Algorithm,* Springer, 2010.

14 C. Yuansi, An Almost Constant Lower Bound of the Isoperimetric Co-efficient in the KLS Conjecture, *Geometric and Functional Analysis* 31 (2021), 34–61.

15 W. Zhao, F. Wang, X. Gao and H. Li, Bondage number of the strong product of two trees, *Discrete Applied Mathematics* 230 (2017), 133–145.

16 commons.wikimedia.org, Erdős–Faber–Lovász conjecture, David Eppstein.

17 commons.wikimedia.org, Kneser graph KG(5,2), Tilman Piesk.