Introduction to Geometry

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Credits: 3 EC
Code: 5122INME3Y

Chapter 1

Euclidean Geometry

1.1 Synthetic geometry

1.1.1 History

Greek Mathematics took off in the sixth century BC when Thales of Miletus visited Egypt and learned about the methods that the Pharaoh's surveyors used to measure the shapes and areas of the fertile lands around the Nile. Thales introduced these methods to the Greek but put a different spin on them. Instead of practical tricks for calculation, he considered them tools to explore a more abstract theory about the nature of space. Starting with some basic assumptions he wove an intricate web of reasoning that uncovered eternal truths about points, lines, triangles and other shapes. This new way of thinking marked the birth of the notion of a mathematical proof and it captured the imagination of many Greek philosophers. In the following two centuries they discovered and proved many well known results in geometry.

The proofs of these theorems rest on other more basic results. To avoid infinite regression or circular reasoning it is important to start with some fixed basic principles that we assume without questioning and use as foundations on which we build the rest. This idea was worked systematically by Euclid.

Although we do not know much about Euclid's life, he most likely taught in Alexandria at the beginning of the third century BC. This city in Egypt, founded by Alexander the Great, hosted the Museum, which could best be described as some kind of university avant la lettre. While teaching in Alexandria, Euclid wrote his magnum opus, *The Elements*, a work of 13 books, which covered most of the mathematics that was known by the Greek at that time.

1.1.2 Axioms

Euclid's basic idea was simple: he started off with a few common notions such as points, circles and lines and some basic postulates about what you could do with them. From this he derived basic properties of these figures. Many of them seemed fairly obvious, but he tried to justify them with the aid of diagrams and using only the postulates he started with. Once he had established these basic properties, he used them to prove more complicated things, which formed the basis for even more difficult statements and so on.

Question 1.1. Look at the beginning of the first chapter of the Elements `https://mathcs.clarku.edu/~djoyce/elements/bookI/bookI.html`. What are the main differences with our way of doing mathematics?

ANSWER 1.1

According to our modern standards the foundations of Euclid's elements are quite shoddy. The five main postulates (axioms in modern terminology) are not sufficient. Already in the first proposition the proof uses something that is not justified by the postulates. To construct an equilateral triangle with a give base Euclid draws two circles with centers the endpoints of the base and radius the length of the base. This can be done by the third postulate. Then he looks at an intersection point of the two circles but it does not follow from the postulates that these circles intersect. This and other problems were only solved by the end of the nineteenth century when Hilbert produced the first full set of axioms for the Euclidean plane.

Question 1.2. Look at Hilbert's set of axioms `http://www.msc.uky.edu/droyster/courses/fall11/MA341/axioms/Hilbert.htm`. Which axiom is crucial to fix the proof of Euclid's first proposition?

ANSWER 1.2

1.1.3 Congruence

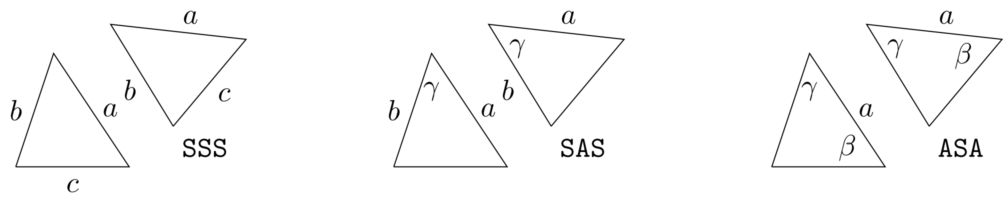
Writing out a proof in terms of Hilbert's axioms is often quite tedious. Instead a lot of proofs in Euclidean geometry rely on two basic techniques: F- and Z-angles and congruence.

Question 1.3. State the idea behind F- and Z-angles and discuss its relation with the fifth postulate and Playfair's axiom (Axiom of Parallels).

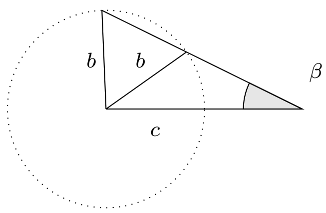
ANSWER 1.3

The concept congruence is not explicitly mentioned by Euclid, but frequently used in disguise. Two figures in the plane are congruent if they have the same shape and size. Obviously if two triangles are congruent, they must have the same sides and angles but Euclid shows us that if you want to check whether two triangles are congruent you do not need to check whether all three sides and angles match up, it suffices to check only

- `SSS` : the three sides or
- `ASA` : one side and two angles or
- `SAS` : one angle and the two adjacent sides.



Note that ASA and SAA are the same criterion because if you know two angles of a triangle you know all three as their sum equals 180°. On the other hand SSA is different from SAS and it does not imply congruence. In general there can be different triangles with the same SSA.



However if the angle is 90° there is only one possibility by Pythagoras's theorem. Therefore SS90° or SSR is sometimes mentioned as the fourth congruence condition.

Question 1.4. In Euclid's Elements the F-and Z-angles correspond to Book I proposition 29. Look up in the Elements which propositions correspond to these three congruence theorems.

ANSWER 1.4

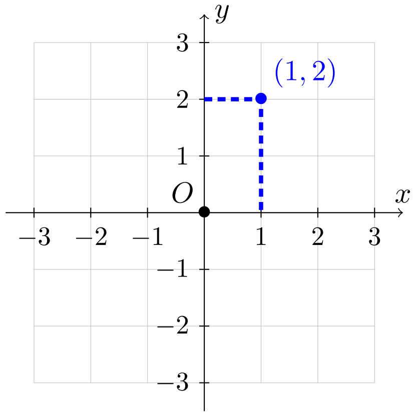
Question 1.5. An important theorem from Euclidean geometry is that the perpendicular bisectors of a triangle are concurrent. Write down a proof using congruence and draw a picture of it.

ANSWER 1.5

1.2 Analytic geometry

1.2.1 Descartes's Idea

In 1637 Rene Descartes published his philosophical magnmum opus *Discours de la Méthode*. Attached to it was an appendix called *La Géométrie* in which he introduced a new way of doing geometry using coordinates. The idea was to fix an origin O and two axes through it. These could be used to give every point a pair of coordinates (its distances from the origin in the direction of the two axes).



Assigning coordinates to a point.

The idea of coordinates had been known since the Greek, but the innovation of Descartes was that lines, circles and other curves in the plane could be represented by equations that described the relations between the $X-$ and Y-coordinate of their points. In this way geometrical statements could be translated into algebraic ones and vice versa.

Question 1.6. Let P, Q be two points with coordinates $(x_P, y_P), (x_Q, y_Q)$. Write down the equation for

(a) The line between P, Q,

(b) The perpendicular bisector of the line segment PQ,

(c) The circle with centre P and radius r,

ANSWER 1.6

1.2.2 The cartesian model

From a modern point of view we can take the opposite approach: instead of starting from the Euclidean plane and then construct coordinates, we can use pairs of numbers to define the Euclidean plane itself. Set $\mathbb{E}^2 := \mathbb{R}^2$ and define

- Points as elements of $\mathbb{E}^2$:
$$P = (x, y) \in \mathbb{R}^2,$$
- Lines as subsets of $\mathbb{E}^2$ that satisfy a linear equation:
$$\ell = \{(x, y) \in \mathbb{R}^2 \mid ax + by + c = 0\},$$
- Distances between points by the Pythagorean formula:
$$d(P, Q) = |PQ| = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2},$$

- Circles as subsets of points that lie at a distance r of a point (x_P, y_P):

$$C = \{(x, y) \in \mathbb{R}^2 \mid (X - x_P)^2 + (Y - y_P)^2 = r^2\},$$

- Angles using the law of cosines:

$$\widehat{PQR} = \arccos \frac{|PR|^2 - |PQ|^2 - |QR|^2}{2|PQ||PR|}.$$

This is the Cartesian model of Euclidean geometry. This approach has the advantage that we do not need to come up with a full set of axioms for the Euclidean plane. The axioms are just consequences of the definitions and the properties of the real numbers.

Question 1.7. Prove Playfair's axiom in the Cartesian model.

ANSWER 1.7

Proving theorems in analytic geometry is usually more straightforward than in synthetic geometry. You just start to calculate your way through the construction to find algebraic expressions for the relevant objects. This often leads to a lot of tedious calculations and therefore it is important to choose a good coordinate system.

Question 1.8. Give an analytic proof of the theorem that the perpendicular bisectors of a triangle are concurrent. Choose a good coordinate system for the proof. Which method do you prefer, synthetic or analytic?

ANSWER 1.8

1.3 Further practice

Exercise 1.1. Prove the following statements synthetically and/or analytically.

(a) The diagonals of a parallelogram bisect each other. (i.e. their intersection point is the midpoint of the diagonals)

(b) The diagonals of a rhombus are perpendicular.

(c) The base angles of an isosceles trapezoid are equal (A quadrangle $ABCD$ with $AB//CD$ is an isosceles trapezoid if $BC = DA$. You also may assume that $AB < CD$ because otherwise it is a parallelogram)

(d) The line connecting the midpoints of two sides of a triangle is parallel with the third side.

(e) Show that the tangent line through a point on a circle stands perpendicular to a radius through that point.

(f) Let P be a point outside a circle and A, B be the two points on the circle such that PA and PB are tangent to the circle. Show that the lengths of PA and PB are the same.

(g) If you put squares on the sides of a parallelogram then the midpoints of the squares are the corners of a square.

(h) If A,B are two points then the set of all points C for which $\triangle ABC$ is a right angled triangle with right angle at C, lie on a circle.

(i) If A, B, C are points on a circle with center P then $\widehat{BAC} = \frac{1}{2}\widehat{BPC}$ (provided P and A lie on the same side of BC. What changes when P and A lie on different sides of BC?.

(j) Show that the medians of a triangle divide the triangle in six parts of equal area.

Exercise 1.2. There are four concurrency theorems for a triangle

(a) In a triangle the three perpendicular bisectors are concurrent.

(b) In a triangle the three angle bisectors are concurrent.

(c) In a triangle the three medians are concurrent.

(d) In a triangle the three altitudes are concurrent.

Try to prove these synthetically and/or analytically. Which do you find the most difficult/tedious to prove and why?

Instageom 1. Take your favourite theorem in Euclidean geometry and illustrate its proof in an artistic or unusual way (in the style of Byrnes: `https://www.c82.net/euclid/`, using everyday objects, using origami, use a real ruler and compass, etc). For a collection of theorems and results `http://www.cut-the-knot.org/geometry.shtml`.

Chapter 2

Conic Sections

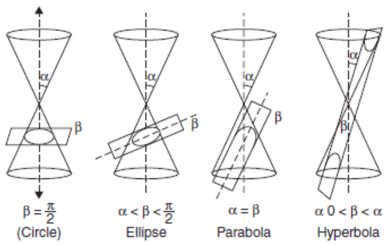
2.1 Definition

Conic sections were known by the Greek mathematicians even before Euclid, but they were not treated in his Elements. The standard classical treatment that survived until today was by Appolonius, who lived in the third century BC and had spent some time in Alexandria. He came up with several geometric properties of these curves. Pappus, who also lived in Alexandria but 500 year later and Kepler, who lived in the 16th century, added some extra characterizations of these curves.

The basic idea is simple. Consider a line in three-dimensional space through the origin that makes an angle α with the z-axis and rotate it around that axis. This traces a cone. Now take a plane that makes an angle β with the z-axis not through the origin and look at the intersection. This is called a (nondegenerate) *conic section*. In particular we call it a

- Ellipse if $\beta > \alpha$.
- Parabola if $\beta = \alpha$.
- Hyperbola if $\beta < \alpha$.

Note that if $\beta = \frac{\pi}{2}$ we get a circle, so this is a special kind of ellipse.



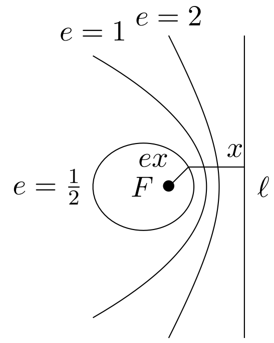
Question 2.1. If the plane goes through the origin we get *degenerate conic sections*. What are the three possibilities in that case?

ANSWER 2.1

2.2 Focus and directrix

2.2.1 A new definition

A second way to describe all three conics sections is due to Pappus. He started with a line ℓ and a point F and looked at the locus of all points for which the distance to that point and the that line have a fixed ratio. This ratio e is called the *eccentricity*, the line the *directrix* and the point the *focus*. If the eccentricity is smaller than one, i.e. the point is always closer than the line, you get an ellipse. If the eccentricity is equal to one you get a parabola and if it is bigger than one you get a hyperbola.



2.2.2 Standard forms

By choosing a good coordinate system we can express these characterizations in an equation.

- If $e < 1$ we put $F = (ae, 0)$ and $\ell : x = \frac{a}{e}$. The equation becomes
$$(x - ea)^2 + y^2 = e^2(x - \frac{a}{e})^2 \rightsquigarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ with } b = a\sqrt{1 - e^2}.$$
This is the standard form of an ellipse.

- If $e > 1$ we again put $F = (ae, 0)$ and $\ell : x = \frac{a}{e}$. The equation becomes
$$(x - ea)^2 + y^2 = e^2(x - \frac{a}{e})^2 \rightsquigarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ with } b = a\sqrt{e^2 - 1}.$$
This is the standard form of a hyperbola.

- if $e = 1$ we put $F = (a, 0)$ and $\ell : x = -a$ and then the equation becomes
$$y^2 = 4ax.$$
This is the standard form of a parabola.

Question 2.2. Work out the case of the parabola.

ANSWER 2.2

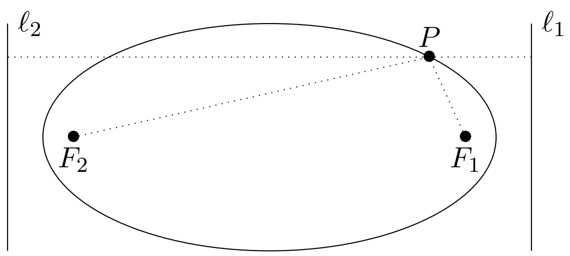
The ellipse and the hyperbola are symmetric about the x- and y-axes in this coordinate system. These two axes are called the major and the minor axis of the ellipse/hyperbola. The origin is also a center of symmetry, it is called the center of the ellipse/hyperbola. The parabola is only symmetric about the x-axis, this is called the axis of the parabola.

2.2.3 The distance to the foci

Note that both the ellipse and the hyperbola have two foci and directrices because the construction with $F = (-ea, 0)$ and $\ell : x = -\frac{a}{e}$ gives the same equation. These two foci can be used for a second property of the ellipse and hyperbola:

- An ellipse is the locus of all points for which which the sum of the distances to the two foci is $2a$.
- A hyperbola is the locus of all points for which the absolute value of the difference of the distances to the two foci is $2a$.

The proof for the ellipse can easily be seen from the picture



$$|PF_1| + |PF_2| = ed(P, \ell_1) + ed(P, \ell_2) = ed(\ell_1, \ell_2) = e2\frac{a}{e} = 2a.$$

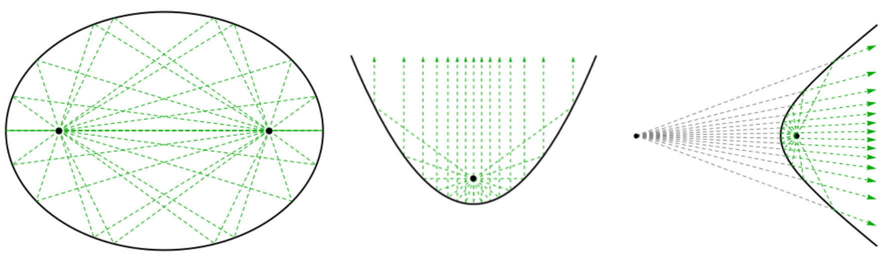
Question 2.3. Adapt the proof to the case of the hyperbola.

ANSWER 2.3

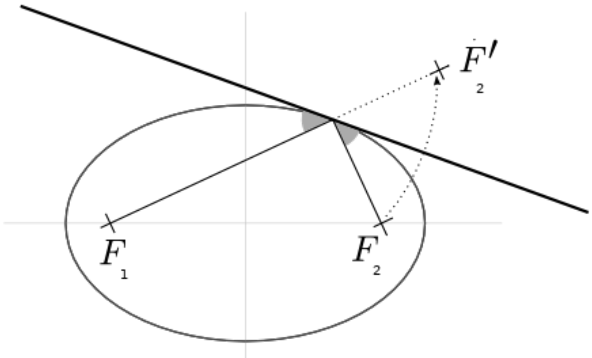
2.2.4 Reflective properties

Finally, the foci also tell us something about the reflective properties of these curves.

- An elliptic mirror will reflect rays emanating from one focus towards the other focus.
- A hyperbolic mirror will reflect rays emanating from one focus as if they emanate from the other focus.
- A parabolic mirror will reflect rays emanating the focus to rays perpendicular to the directrix.



Again we prove the case of the ellipse pictorially. Let P be a point on the ellipse. Reflect the second focus F_2 about the tangent line through P and denote the reflection by F_2'. We have to show that F_1, P and F_2' lie on a straight line.



If this were not the case PF_1F_2' would form a triangle. Denote the intersection point of the tangent line with F_1F_2' by Q. We have that

$$|F_1Q| + |F_2Q| = |F_1Q| + |F_2'Q| = |F_1F_2'| < |F_1P| + |F_2'P| = |F_1P| + |F_2P| = 2a,$$

where $|F_1F_2'| < |F_1P| + |F_2'P|$ follows from the triangle inequality in $\triangle PF_1F_2'$. The inequality $|F_1Q| + |F_2Q| < 2a$ implies that Q must lie inside the ellipse but this is impossible because it lies on the tangent line.

Question 2.4. Work out the case for the parabola.

ANSWER 2.4

Question 2.5. The reflective properties of conic sections have a lot of practical and even legendary applications. Can you find some?

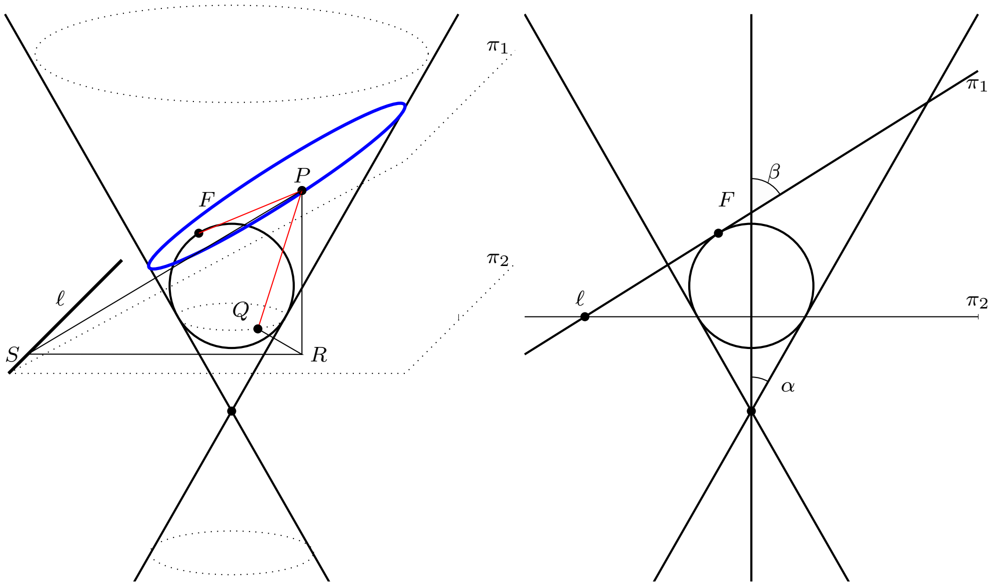
ANSWER 2.5

2.3 Equivalence

We have seen different characterizations of conic sections in terms of foci and directrices but what is lacking is a proof that these characterizations correspond to conic sections in their original definition: intersections of a cone and a plane.

2.3.1 Dandelin spheres

This can be done synthetically using an idea by a Belgian mathematician from the beginning of the 19th century: Germinal Dandelin. Consider the sphere that is tangent to the cone and the plane π_1 that contains the conic section. This sphere touches the plane in a point F and the cone in a circle. This circle lies in a second plane π_2, which is horizontal. Denote the line of intersection of the two planes by ℓ.



With these definitions fixed we will now show that

- ℓ is the directrix,
- F is the focus,
- $e = \frac{\cos\beta}{\cos\alpha}$ is the eccentricity.

Let P be a point on the conic section and PF and PQ be two tangent lines from P to the sphere; PF in the plane π_1 and PQ along the cone. Project P perpendicularly onto the directrix to S and vertically onto the horizontal plane π_2 to R. Then $\triangle PQR$ is a right-angled triangle with $\widehat{P} = \alpha$ and $\triangle PRS$ is a right-angled triangle with $\widehat{P} = \beta$. Therefore

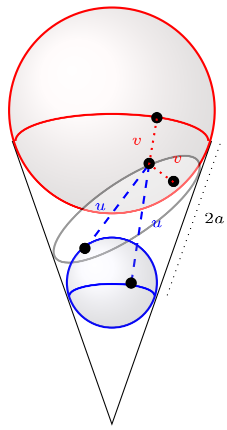
$$|PF| = |PQ| = \frac{|PR|}{\cos\alpha} = \frac{\cos\beta}{\cos\alpha}|PS|.$$

So the distance from P to the focus F is $e = \frac{\cos\beta}{\cos\alpha}$ times the distance from P to ℓ.

Question 2.6. Explain why the sphere used in the proof exists. How do you construct its center and radius?

ANSWER 2.6

Question 2.7. Prove the sum-distance property of the ellipse using the same technique with two Dandelin spheres as indicated in the picture below



ANSWER 2.7

2.3.2 An analytic approach

Another approach you can take involves analytic geometry. The equation of the cone is $x^2 + y^2 = (\tan\alpha)^2 z^2$. This is a quadratic equation in x, y, z. If we embed a plane into 3d-space we can express the (x, y, z)-coordinates of its points in terms of the (x, y)-coordinates of the plane. These expressions are linear and therefore the equation for the intersection between the cone and the plane will be a quadratic equation in (x, y). If we can show that every quadratic equation defines a conic section we are done.

2.4 Quadratic equations

The study of curves defined by quadratic equations in the plane was first done by Fermat and Descartes, the two founding fathers of analytical geometry. Suppose that we have an equation

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

then we can bring it in one of the standard forms of a conic sections with the following steps

- Write the equation as

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} A & \frac{C}{2} \\ \frac{C}{2} & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} D & E \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + F = 0$$

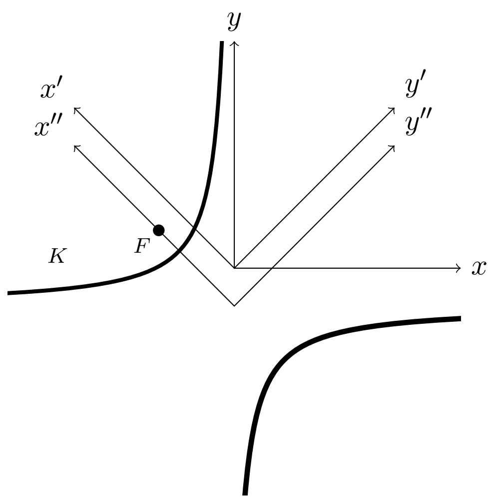
- Find two orthogonal eigenvectors of $\begin{pmatrix} A & \frac{C}{2} \\ \frac{C}{2} & B \end{pmatrix}$ with norm 1. (Remember that the eigenvectors of a symmetric matrix can be chosen orthogonal).

- Choose a coordinate system with axes in the directions of the eigenvectors of the 2×2-matrix to make $C = 0$. In the new equation A, B are the eigenvalues of the matrix and because the original equation was quadratic we can assume that in the new equation we have $A \neq 0$.

- If $B \neq 0$ shift the origin to $(\frac{D}{2A}, \frac{E}{2B})$ (this is the center of the conic section) to make $D, E = 0$. The new equation becomes $Ax^2 + By^2 + F = 0$. If $F = 0$ we get one point or two lines. Otherwise rescale F to -1. This gives the empty set, an ellipse or hyperbola depending on the signs of A, B.

- If $B = 0$ shift the origin to $(\frac{D}{2A}, 0)$. The equation becomes $AX^2 + EY + F = 0$, which is a parabola (or two lines if $E = 0$).

Example 2.4.1. Let us work this out for the example $K : 2xy + x + 1 = 0$.



The eigenvectors of the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are $\begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$, $\begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$ with eigenvalues $-1, 1$. If we do the coordinate transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = x' \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} + y' \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}.$$

The equation becomes

$$-x'^2 + y'^2 - \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' + 1 = 0$$

If we shift the coordinate system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x'' - \frac{\sqrt{2}}{4} \\ y'' - \frac{\sqrt{2}}{4} \end{pmatrix}.$$

The equation becomes

$$-x''^2 + y''^2 + 1 - (-\frac{\sqrt{2}}{4})^2 + (-\frac{\sqrt{2}}{4})^2 = 0 \rightsquigarrow x''^2 - y''^2 = 1$$

This is a hyperbola with eccentricity $\sqrt{2}$, and focus $(x'', y'') = (\sqrt{2}, 0)$. In the original coordinates this means

$$\begin{aligned} F : (x', y') &= (\sqrt{2} - \frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}) \\ &= \sqrt{2}(\frac{3}{4}, -\frac{1}{4}) \\ (x, y) &= (-1, \frac{1}{2}). \end{aligned}$$

Question 2.8. Give a condition in terms of A, B, C for when the conic section is an ellipse, parabola or hyperbola.

ANSWER 2.8

2.5 Further Practice

Exercise 2.1. Consider the points $P_1 = (0, 0)$, $P_2 = (1, 0)$, $P_3 = (1, 1)$ and $P_4 = (0, 1)$ in the Euclidean plane. Find all conic sections through these 4 points. Describe their type and draw them.

Exercise 2.2. Classify these conic sections, bring them in standard form and determine eccentricity and focus.

(a) $3x^2 - 8xy + 3y^2 - 2x + 4y - 16 = 0$,

(b) $x^2 + 8xy + 16y^2 - x + 8y - 12 = 0$,

(c) $52x^2 - 72xy + 73y^2 - 32x - 74y + 28 = 0$.

Exercise 2.3. Show the focus and directrix property for the parabola using a Dandelin sphere.

Exercise 2.4. Determine the ratio between the radii of the two Dandelin spheres of an ellipse in terms of α, β.

Exercise 2.5. Let K be the conic section containing $\{(0, 0), (0, 6), (8, 0), (4, -2), (-1, 3)\}$. What type is it? Give an equation in the form

$$\left(\frac{x - x_0}{a}\right)^2 \pm \left(\frac{y - y_0}{b}\right)^2 = 1$$

with $a, b \in \mathbb{R}$ and $(x_0, y_0) \in \mathbb{R}^2$ the center. Determine the eccentricity.

Exercise 2.6. Let $f(x, y)$ be a quadratic function that reaches a maximum at (u, v). Show that $f(x, y) = 0$ determines an ellipse with center (u, v).

Instageom 2. Conic sections appear in many aspects of everyday life (trajectories of objects, shadows, shapes of bridges, etc). Take a picture of a conic section and give a little explanation of why it appears in this situation. Pinpoint the focus and directrix on the picture. Draw a hyperbola, parabola or ellipse using a string, a ruler and pins and share your artwork.

Chapter 3

Symmetries and Affine geometry

3.1 Isometries

3.1.1 Definition

Remember that we defined the Euclidean plane as $\mathbb{E}^2 = \mathbb{R}^2$ and represent a point P by a pair (x_P, y_P). On $\mathbb{E}^2$, we define a distance function $d(p,q) = |p-q| = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}$. An *isometry* of the Euclidean plane is a bijective map

$$\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$$

that preserves distances:

$$\forall p, q \in \mathbb{R}^2 : d(p,q) = d(\varphi(p), \varphi(q)).$$

Question 3.1. Show that the set of isometries $\mathsf{Isom}\,\mathbb{E}^2$ forms a group for the composition.

ANSWER 3.1

3.1.2 Algebraic description

An isometry φ is completely determined by the images of three points:

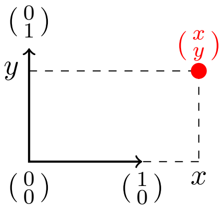
$$\varphi\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \quad \varphi\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \quad \varphi\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right).$$

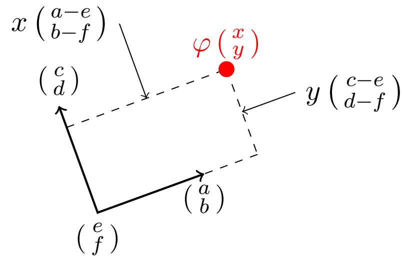
Indeed, by the picture below, we can see that if

$$\varphi\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}e\\f\end{smallmatrix}\right), \varphi\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}a\\b\end{smallmatrix}\right) \text{ and } \varphi\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}c\\d\end{smallmatrix}\right)$$

then

$$\varphi\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = \underbrace{\left(\begin{smallmatrix}a-e & c-e\\b-f & d-f\end{smallmatrix}\right)}_{A}\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) + \underbrace{\left(\begin{smallmatrix}e\\f\end{smallmatrix}\right)}_{B}$$





Not every map of the form

$$\varphi : \left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \mapsto A\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) + B$$

is an isometry. For this to hold we need an extra condition.

Question 3.2. Show that φ is an isometry if and only if $A^\top A = 1$ (such a matrix is called orthogonal because the columns of A are orthogonal and have length one).

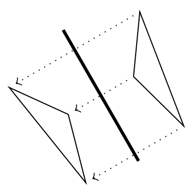
ANSWER 3.2

There are 2 possibilities for A if $A^\top A = 1$: its determinant can be $+1$ or -1. If $\det A = 1$ we call the isometry *direct* and otherwise we call it *indirect*. By parametrizing the first column of A as $\left(\begin{smallmatrix}\cos\theta\\ \sin\theta\end{smallmatrix}\right)$ we obtain that every isometry of the Euclidean plane can be written uniquely as one of these two forms

- $\varphi\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = \left(\begin{smallmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{smallmatrix}\right)\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) + B$ if it is a direct isometry.
- $\varphi\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = \left(\begin{smallmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{smallmatrix}\right)\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) + B$ if it is an indirect isometry.

3.1.3 Geometric description

We can also describe isometries geometrically. The basic building blocks for this point of view are the reflections. A *reflection* σ_ℓ is a nontrivial isometry that fixes all points on a line ℓ, which is called the axis. Note that for all points $P \notin \ell$ the axis is the perpendicular bisector of P and $\sigma_\ell(P)$.



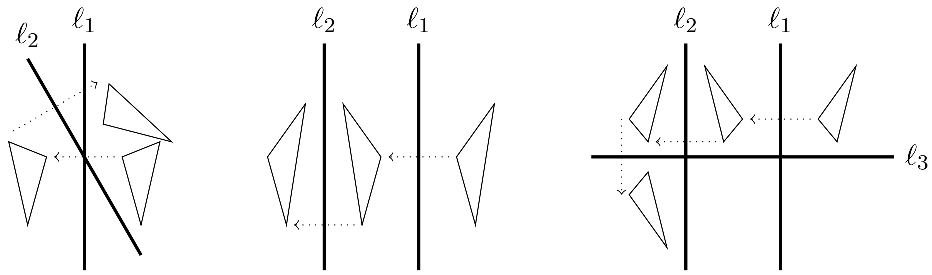
Every isometry of the Euclidean plane is the composition of at most 3 reflections. If $P_0 = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$, $P_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, $P_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ and $\varphi(P_i) = Q_i$ then we can make φ in the following way. First reflect about the perpendicular bisector of P_0Q_0. This maps P_0 to Q_0. Denote the images of P_i after this first reflection P_i'. If $P_1' \neq Q_1$ reflect about the perpendicular bisector of $P_1'Q_1$. Note that Q_0 lies on the axis of this reflection because $|P_1'Q_0| = |Q_1Q_0|$. Denote the images of P_i' after this second reflection P_i''. We have that $P_0'' = Q_0$ and $P_1'' = Q_1$. Finally if $P_2'' \neq Q_2$ reflect about Q_0Q_1 to map P_2'' onto Q_2.

Question 3.3. Explain why Q_0Q_1 is the perpendicular bisector of $P_2''Q_2$.

ANSWER 3.3

Depending on how the axes of the reflections relate to each other we can distinguish three other types of isometries.

- *Rotations.* These are compositions of two reflections with intersecting axes. The intersection point is the center of the rotation and the angle is twice the angle between the axes.
- *Translations.* These are compositions of two reflections with parallel axes. The distance of the translation is twice the distance between the axes.
- *Glide reflections.* These are compositions of three reflections that cannot be written as a composition of fewer reflections.



To determine what type a given map $\varphi : \mathbf{x} \mapsto A\mathbf{x} + B$ is one can do the following

- If $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ it is a translation over a vector B.
- If $A = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ it is a rotation over an angle θ. The center of rotation is the unique fix point $\varphi(\mathbf{x}) = \mathbf{x}$.
- If $A = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right)$, solve the equation $\varphi(\mathbf{x}) = \mathbf{x}$. If there are no solutions then it is a glide reflection; if the solutions form a line it is a reflection about that line.

Question 3.4. Show that in the last case the reflection axis makes an angle $\frac{\theta}{2}$ with the X-axis.

ANSWER 3.4

3.2 Euclidean geometry

3.2.1 Congruence

Two figures (subsets of the Euclidean plane $\mathbb{E}^2$) are *congruent* if there is an isometry that maps one to the other.

Question 3.5. Show congruence is an equivalence relation.

ANSWER 3.5

Two line segments $[PQ]$ and $[P'Q']$ are congruent if they have the same length because it is easy to find an isometry between them: first shift P to P' and then rotate around P' to match the second end points. In a similar way one can show that two triangles are congruent if one of the three congruence conditions holds (SSS/SAS/SAA). Circles are congruent if they have the same radius.

3.2.2 Euclidean concepts

Euclidean geometry studies concepts and properties that are preserved under isometries. Examples of these are: distances, angles, straight line, circle. Some concepts are not preserved under isometries: horizontal, clockwise, ...

Question 3.6. Give more examples of concepts that are preserved and are not preserved under isometries.

ANSWER 3.6

Usually theorems in Euclidean geometry come with a diagram. If you apply an isometry to this diagram, you should get an equally valid diagram for the theorem. E.g. for the proof that the base angles of an isosceles triangle are equal, it does not matter whether in the diagram the triangle points upwards or downwards.

Objects that are especially interesting are those that possess symmetries: there are isometries that map the objects to itself. An equilateral triangle is invariant under three rotations and three reflections. A circle is invariant under an infinite number of rotations and a line is invariant under translations.

Question 3.7. In Euclidean geometry we make a distinction between parallelograms, rhombi, rectangles, and squares. Describe the differences between them in terms of symmetries.

ANSWER 3.7

3.2.3 Active and passive

We can use symmetries in two ways: actively and passively.

- An active transformation moves all the points in the plane: a point $\mathbf{x} \in \mathbb{E}^2$ is mapped to the point $A\mathbf{x} + B$.
- A passive transformation keeps all the points fixed but moves the coordinate system. This means that if a point has coordinates $\mathbf{x}'$ in the new coordinate system it will have had coordinates $\mathbf{x} = A\mathbf{x}' + B$ in the old coordinate system.

Therefore from the point of view of coordinates an active transformation over φ is the same as a passive transformation over φ^{-1} and vice versa. This is important if we work with equations. If $C \subset \mathbb{E}^2$ is a curve given by the equation $f(\mathbf{x}) = 0$ and $\varphi \colon \mathbb{E}^2 \to \mathbb{E}^2$ is a transformation then the curve $\varphi(C) := \{\varphi(\mathbf{x}) \mid f(\mathbf{x}) = 0\}$ will be described by the equation $f(\varphi^{-1}(\mathbf{x})) = 0$.

3.3 Affine transformations

3.3.1 Definition

We will now broaden the concept of symmetry to allow more general transformations. An *affine transformation* is a map of the form

$$\varphi \colon \mathbb{R}^2 \to \mathbb{R}^2 \colon \mathbf{x} \mapsto A\mathbf{x} + B$$

where A is an invertible 2×2-matrix and $B \in \mathbb{R}^2$. Just like the isometries, the affine transformations form a group which we denote by $\mathrm{Aff}(2, \mathbb{R})$.

Question 3.8. Show that the inverse of the map $\mathbf{x} \mapsto A\mathbf{x} + B$ is again an affine transformation. What are its A and B?

ANSWER 3.8

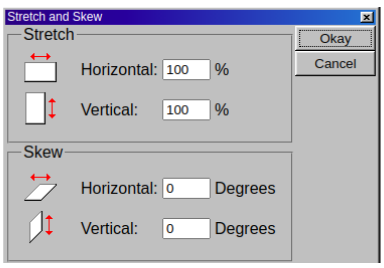
An affine transformation maps straight lines to straight lines. Indeed you can check that $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ lies on the line $\ell : ux + vy + w = 0$ if and only if $\varphi\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ lies on the line $\ell' : u'x + v'y + w' = 0$ where

$$\left(\begin{smallmatrix} u' & v' \end{smallmatrix}\right) = \left(\begin{smallmatrix} u & v \end{smallmatrix}\right) A^{-1} \text{ and } w' = w - \left(\begin{smallmatrix} u & v \end{smallmatrix}\right) A^{-1} B.$$

3.3.2 Stretches and skews

Every affine transformation is the composition of a map of the form $\varphi_A \colon \mathbb{R}^2 \to \mathbb{R}^2 \colon \mathbf{x} \mapsto A\mathbf{x}$ followed by a translation over B. Therefore we will concentrate on maps of the form φ_A. We already encountered rotations and reflections but there are more possibilities.

Question 3.9. Stretches and skews are two types of affine transformations.



Give the corresponding matrices of these transformations.

ANSWER 3.9

Every invertible matrix can be written as a product of matrices of these types and therefore every affine transformation can be made from translations, skews and stretches (possibly with a negative stretch factor).

3.4 Affine Geometry

3.4.1 Affine congruence

Two figures are *affine congruent* if there is an affine transformation that maps one to the other. Because every Euclidean transformation is affine, two objects that are Euclidean congruent are also affine congruent. The converse is not true.

Take for instance triangles. All triangles are affine congruent to the standard triangle with points $\binom{0}{0}$, $\binom{1}{0}$, $\binom{0}{1}$ because if $P_0 = \binom{e}{f}$, $P_1 = \binom{a}{b}$, $P_2 = \binom{c}{d}$ then the affine transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a-e & c-e \\ b-f & d-f \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

maps the standard triangle to $\triangle P_0 P_1 P_2$. Note that this affine transformation is uniquely determined.

We can also have a look at congruences for conic sections. We have already seen using Euclidean transformations we can bring each conic section into one of the standard forms. For an ellipse this is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Ellipses with different a, b are not Euclidean congruent but we can stretch the horizontal and vertical directions

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

such that $(a, 0) \mapsto (1, 0)$ and $(0, b) \mapsto (0, 1)$. After this transformation the equation becomes $x^2 + y^2 = 1$, so every ellipse is affine congruent to the unit circle. In a similar way, every hyperbola is affine congruent to the one with equation $x^2 - y^2 = 1$, and every parabola to the one with equation $y^2 = x$.

3.4.2 Affine concepts and theorems

Affine geometry studies concepts and objects that are preserved under affine transformations. Examples of these are: straight line, parallel, triangle, the midpoint of a line segment. Some concepts are not preserved under affine transformations: perpendicular, rhombus, circle.

Question 3.10. Give more examples of concepts that are preserved and are not preserved under affine transformations.

ANSWER 3.10

Theorems in affine geometry should only deal with concepts that are preserved under affine transformations. Moreover if you apply an affine transformation to a diagram for the theorem, you should get an equally valid diagram for the theorem.

For example the theorem that the medians of a triangle are concurrent is an affine theorem because affine transformations preserve midpoints. The theorem that the altitudes of a triangle are concurrent is not an affine theorem because affine transformations do not preserve perpendicularity.

Question 3.11. Give some examples of classical theorems in Euclidean geometry that are in fact affine theorems.

ANSWER 3.11

If we are doing affine geometry instead of Euclidean geometry, we call the plane in which we work the affine plane and denote it by $\mathbb{A}^2$. As a set $\mathbb{E}^2$ and $\mathbb{A}^2$ are both equal to $\mathbb{R}^2$ but the latter has more symmetries than the former.

3.5 Further practice

Exercise 3.1. Give the transformations $\mathbf{x} \mapsto A\mathbf{x} + B$ that correspond to the following isometries

(a) a reflection about the line $x + y + 2 = 0$,

(b) a rotation around the point $(1, -1)$ over $90°$,

(c) a rotation around the point $(1, 0)$ over $60°$,

(d) a reflection about the line $x + 2y + 1 = 0$.

Exercise 3.2. Indicate whether the following concepts in Euclidean geometry terms still exist in affine geometry. Please explain your answer.

(a) chordal quadrilateral (koordenvierhoek),

(b) center of gravity of a triangle,

(c) straight angle (gestrekte hoek),

(d) bisector of an angle,

(e) tangent line to an ellipse.

Exercise 3.3. Given is the following conic section:

$$x^2 + 6xy + y^2 - 2\sqrt{2}x - 2\sqrt{2}y = 0.$$

(a) Find an Euclidean transformation that brings this conic into its standard shape.

(b) Give an affine transformation that maps this conic to the conic described by

$$xy = 1.$$

Exercise 3.4. [Exercise 1 and 2 of H2 van Brannan, p86]

(a) Find the image of the line

$$3x - y + 1 = 0$$

under the affine transformation given by

$$t(\mathbf{x}) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -\frac{3}{2} \\ 4 \end{pmatrix}.$$

(b) Find the image of the circle

$$x^2 + y^2 = 1$$

under the affine transformation given by

$$t(\mathbf{x}) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -\frac{3}{2} \\ 4 \end{pmatrix}.$$

Exercise 3.5. The medians of a triangle are the lines that pass through a vertex X and the center of the opposite side M_X.

(a) Show that the medians of any triangle go through one point. This point is called the center of gravity of the triangle.

(b) Let G be the center of gravity of triangle $\triangle ABC$. Show that

$$\frac{AG}{GM_A} = \frac{BG}{GM_B} = \frac{CG}{GM_C} = 2.$$

Explain why these theorems are affine theorems.

Exercise 3.6. Find all affine transformations that hold the origin $(0, 0)$ and map the parabola $y = x^2$ to itself.

Exercise 3.7. It is possible to make a rotation out of skews. If θ is any angle $\alpha = -\frac{\theta}{2}$ and $\beta = \tan^{-1}(\sin(\theta))$. Show that if you first do a horizontal skew over α, then vertically over β and then again horizontally over α, the resulting affine transformation will be a rotation over θ. To show this you need to use the formulas for the sin and cosine of the double of an angle:

$$\sin 2\varphi = 2 \sin \varphi \cos \varphi, \qquad \cos 2\varphi = 1 - 2 \sin^2 \varphi.$$

Exercise 3.8. Let A, B, C be three points on a line and A', B', C' three other points on a line. Show that the triple (A, B, C) is affine congruent with (A', B', C') if and only if

$$\frac{\vec{AB}}{\vec{BC}} = \frac{\vec{A'B'}}{\vec{B'C'}}$$

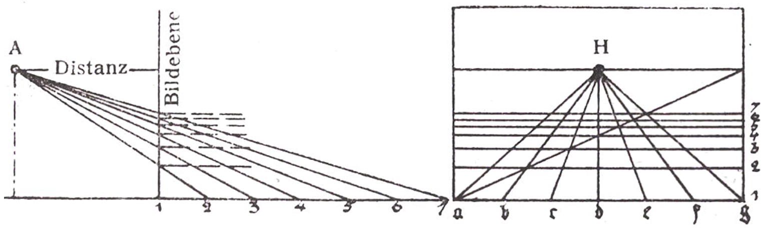
Instageom 3. Take a tiling of the plane and describe its symmetries (the axes of reflections, the centers of rotations). There are 17 different types of symmetry patterns `https://mathworld.wolfram.com/WallpaperGroups.html`. Choose one and try to draw a plane tiling with that pattern. Can you also draw patterns with affine symmetries that are not isometries?

Chapter 4

The Projective plane

4.1 The horizon and points at infinity

In the fifteenth century, Renaissance artists like Brunelleschi and Della Francesca unlocked the mystery of mathematically correct perspective drawing. The key idea is that lines on the ground that are parallel in the real world look as if they intersect on the horizon in the picture.



Question 4.1. Suppose that the eye of the painter is at $(0,0,0)$ and the canvas is at the plane $C : y = 1$. Every point $p = (x,y,z)$ in the three-dimensional space is projected via a straight line through the eye of the painter onto C. Finally, let the ground be given by the plane $G : z = 1$ (so downwards is the positive z-direction). Show that two parallel lines on the ground are projected to lines on the canvas that intersect in a point on the line $h : y = 1, z = 0$.

ANSWER 4.1

Note that every point on the ground plane lies on a unique line through the origin, so it looks as if there is a bijection between lines through the origin and points on the ground plane. However, this is not completely true: there are lines through the origin that are parallel to the ground plane and hence do not intersect it. (Note that these lines go through the horizon on the canvas.) We could solve this by adding extra *points at infinity* to the ground plane, one for each line through the origin that is parallel to the ground plane. All these extra points form a line, which we call the *line at infinity*. This gives rise to the notion of the projective plane.

4.2 The projective plane

The projective plane $\mathbb{RP}^2$ is the set of lines through the origin. Because a line through the origin can be written as $\mathbb{R}\mathbf{v}$ where $\mathbf{v}$ is a nonzero (!) vector, we have

$$\mathbb{RP}^2 = \{\mathbb{R}(x, y, z) \mid (x, y, z) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}\}.$$

We denote the elements of $\mathbb{RP}^2$ by $(x : y : z) = \mathbb{R}(x, y, z)$ or $[\mathbf{v}] = \mathbb{R}\mathbf{v}$ where $\mathbf{v}$ is a nonzero vector. An element $(x : y : z) \in \mathbb{RP}^2$ is also called a *projective point*. Two triples correspond to the same projective point in $\mathbb{RP}^2$ if they differ by a scalar multiple:

$$(x : y : z) = (\lambda x : \lambda y : \lambda z) \text{ or } [\mathbf{x}] = [\lambda \mathbf{x}].$$

A *projective line* in $\mathbb{RP}^2$ is a set of projective points that all lie in the same plane through the origin.

$$\ell = \{(x : y : z) \in \mathbb{RP}^2 \mid ax + by + cz = 0\}.$$

Question 4.2. Show that any two lines in the projective plane always intersect in one point, so in the projective plane there is no notion of parallel lines.

ANSWER 4.2

We can embed the Euclidean plane into the projective plane:

$$\iota \colon \mathbb{R}^2 \to \mathbb{RP}^2 \colon (x, y) \mapsto (x : y : 1).$$

The complement of the image of this embedding is all projective points with $z = 0$. These form a projective line, which is the line at infinity.

If $ax + by + c = 0$ is a line in the Euclidean plane, then its image under ι all lie on the projective line given by the equation $ax + by + cz = 0$. This projective line contains one extra point $(b : -a : 0)$ at infinity.

Question 4.3. Show that if ℓ_1 and ℓ_2 are two parallel lines in the Euclidean plane, the corresponding projective lines go through the same point at infinity.

ANSWER 4.3

If $C = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$ is a curve in the Euclidean plane, we can map it to the projective plane with ι. The points in $\iota(C)$ will satisfy a new equation $F(x, y, z) = 0$, which can be obtained by homogenizing $f(x, y)$: if the total degree of f is n we add powers of z to each monomial in f to make them all have the same degree n;

$$f(x, y) = \sum f_{ij} x^i y^j = 0 \rightsquigarrow F(x, y, z) = \sum f_{ij} x^i y^j z^{n-i-j} = 0.$$

The homogeneous equation has the property that $F(\lambda x, \lambda y, \lambda z) = \lambda^n F(x, y, z)$, so $F(x, y, z) = 0$ for all possible representatives of the point $(x : y : z)$. This makes the set

$$\{(x : y : z) \in \mathbb{RP}^2 \mid F(x, y, z) = 0\} \supset \iota(C)$$

well defined. The process of homogenization adds some extra points at infinity, which are those of the form $(x : y : 0)$.

Question 4.4. Give the homogeneous equations for an ellipse, hyperbola and parabola in standard form. Show that they have $0, 2$ and 1 extra point at infinity, respectively.

ANSWER 4.4

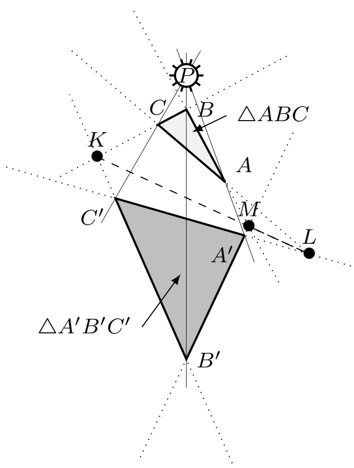
Vice versa, every homogeneous equation $F(x, y, z) = 0$ of degree n defines a well-defined subset of $\mathbb{RP}^2$. If we restrict this equation to the Euclidean plane we get $F(x, y, 1) = 0$.

4.3 Projective theorems

4.3.1 Desargues's theorem

In 1639, the French mathematician Girard Desargues came up with a new theorem that at first sight looks quite complicated because it involves 10 lines and 10 points.

Start with a point P and draw three lines through it. Choose on each line two points and name them A, A', B, B' and C, C'. This configuration gives rise to two triangles $\triangle ABC$ and $\triangle A'B'C'$ and we can look at corresponding sides. Each pair of corresponding lines BC, $B'C'$ have an intersection point. We end up with three new points K, L, M. Desargues showed that these three points lie on one line.



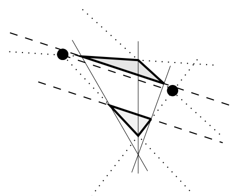
The shady interpretation of Desargues's theorem

This complicated story becomes clearer if we put it in three dimensions. Then the two triangles lie in different planes and the second triangle can be seen as the shadow cast by the first triangle on the second plane under a light source in P. The lines in the second plane are shadows of the lines in the first plane and they must intersect each other in the intersection of the two planes. So all three points K, L, M must lie on the intersection line. This gives an easy proof of the theorem in three dimensions. In two dimensions, life is harder, but Desargues was able to produce a proof that used classical Euclidean geometry to compare lengths of certain line segments.

4.3.2 Unifying Euclidean theorems

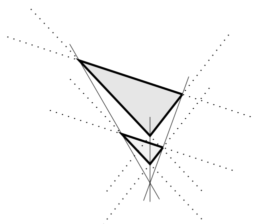
This theorem, nice and clean though it seems, still has its dusty corners. Not every random choice of points on the three lines will do the trick. In some cases, the sides of the triangles are parallel and there is no intersection point K, L or M. These are very special situations but they do occur. Each situation can be patched by a small variant on the main theorem.

For example, if K does not exist but L and M do, a different theorem tells us that the edges BC and $B'C'$ are parallel to the line LM. From a Euclidean point of view this is a new theorem because it now deals with parallel lines, but if we literally put it in perspective something odd happens.



Desargues's theorem with one point at infinity: one pair of sides is parallel to the line through the intersection points of the other pairs

If we lay out the diagram of this new theorem on the floor and look at it from a distance, the two parallel lines will intersect in a vanishing point at the horizon and the line LM, which is also parallel to this line, will intersect at the horizon in the same point. If we call that hypothetical point at the horizon K, the new version of the theorem just a rehash of the old one.



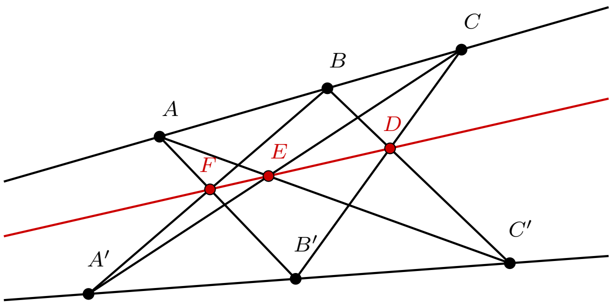
Desargues's theorem with the line at infinity: all corresponding sides are parallel

What happens if both K and L do not exist? In the perspective drawing the corresponding pairs of parallel lines will intersect the horizon in hypothetical points that we also conveniently name K and L. The old version of the theorem says that the point M must lie on the same line as K and L, which is the horizon. This means that in the original diagram M does not exist and all three pairs of lines are parallel. Translating this back we get yet another theorem: if BC and $B'C'$ are parallel and AC and $A'C'$ are parallel then AB and $A'B'$ are also parallel. Using the idea of hypothetical points on the horizon, different theorems in the Euclidean plane can be seen as different perspectives on the same projective theorem.

So we can conclude that by extending the Euclidean plane to the projective plane, it is possible to merge different Euclidean theorems into one projective theorem. To study this in more detail we will look at projective transformations in the next lesson.

Question 4.5. Pappus's theorem states the following:

Consider two lines ℓ_1, ℓ_2 and six points $A, B, C \in \ell_1$, $A', B', C' \in \ell_2$. Let D, E, F be the intersection point between BC' and CB', AC' and CA', and AB' and $A'B$. Then D, E, F lie on a line.



Give two Euclidean variants: one where one of the points lies at infinity and one where one line lies at infinity.

ANSWER 4.5

4.4 Further practice

Exercise 4.1. Determine an equation for each of the following lines in $\mathbb{RP}^2$:

(a) the line through the points $(1 : 2 : 3)$ and $(3 : 0 : -2)$,

(b) the line through the points $(1 : -1 : -1)$ and $(2 : 1 : -3)$.

Exercise 4.2. Determine whether each of the following sets of points are collinear:

(a) $(1 : -1 : 0), (1 : 0 : -1)$ and $(2 : -1 : -1)$,

(b) $(1 : 0 : 1), (0 : 1 : 2)$ and $(1 : 2 : 3)$.

Exercise 4.3. Determine the point of intersection of each of the following pairs of lines in $\mathbb{RP}^2$:

(a) the lines with equations $x - 2y + z = 0$ and $x - y - z = 0$,

(b) the lines with equations $x + 2y + 5z = 0$ and $3x - y + z = 0$.

Exercise 4.4. Give conditions for when the Euclidean curve $Ax^2 + By^2 + \cdots + F = 0$ has $0, 1$ or 2 points at infinity.

Instageom 4. Take a picture of a scene with a lot of straight lines. Draw the points at infinity where these lines intersect. Look up the notions of 1,2,3-point perspective and draw a perspective scene yourself. Find a painting from the 14$^{\text{th}}$ century before perspective drawing was invented and try to find errors in it. Take a picture that uses perspective in a special/surprising way.

Chapter 5

Projective Geometry

5.1 Projective transformations

Now that we have constructed a new kind of space, $\mathbb{RP}^2$, we also want to develop a geometry for this space. Just like affine transformations formed the basis for affine geometry we will need a new type of transformations as the basis for this new geometry.

5.1.1 Definition

A *projective transformation* is a map

$$t_A : \mathbb{RP}^2 \to \mathbb{RP}^2 : [\mathbf{x}] \mapsto [A\mathbf{x}]$$

where A is an invertible 3×3 matrix. Note that A and λA define the same projective map.

Question 5.1. Show that the projective transformations form a group isomorphic to

$$\mathrm{PGL}(3, \mathbb{R}) = \mathrm{GL}(3, \mathbb{R})/\mathbb{R}^*$$

where $\mathrm{GL}(3, \mathbb{R}) = \{A \in \mathrm{Mat}_{3\times 3}(\mathbb{R}) \mid \det A \neq 0\}$.

ANSWER 5.1

Because multiplication with A is a linear map, planes through the origin are mapped to planes through the origin. From the point of view of $\mathbb{RP}^2$ this means that t_A maps projective lines to projective lines. Indeed if ℓ is the line given by the equation

$$ax + by + cz = 0$$

we can write this equation as $\mathbf{a}^\top \mathbf{x} = 0$ with $\mathbf{a}^\top = (a, b, c)$. If $\mathbf{x}$ satisfies the equation $\mathbf{a}^\top \mathbf{x} = 0$ then $\mathbf{y} = A\mathbf{x}$ will satisfy the equation $\mathbf{b}^\top \mathbf{y} = 0$ with $\mathbf{b} = A^{-1\top}\mathbf{a}$. So t_A maps the line with coefficient vector $\mathbf{a}$ to the line with coefficient vector $A^{-1\top}\mathbf{a}$.

If we embed the affine plane $\mathbb{R}^2$ into $\mathbb{RP}^2$ by $(x, y) \mapsto (x : y : 1)$ then we can extend every affine transformation $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ to a projective transformation determined by the matrix

$$\begin{pmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ 0 & 0 & 1 \end{pmatrix}$$

So projective transformations can be seen as generalizations of affine transformations.

5.1.2 Quadrangles

We have already seen that an affine transformation can map every triangle to any other triangle. For projective transformations we have something similar. A projective quadrangle consists of four projective points of which no three lie on a projective line. The standard projective quadrangle consists of

$$P_1 = (1 : 0 : 0),\ P_2 = (0 : 1 : 0),\ P_3 = (0 : 0 : 1),\ P_4 = (1 : 1 : 1).$$

If $\mathbf{u_1}, \dots, \mathbf{u_4}$ are four vectors such that $[\mathbf{u_1}], \dots, [\mathbf{u_4}]$ form a projective quadrangle then there is a unique projective transformation t_A that maps P_i to $[\mathbf{u_i}]$.

To find a matrix A first note that because no three points are collinear $\mathbf{u_1}, \dots, \mathbf{u_3}$ form a basis for $\mathbb{R}^3$ and therefore we can write $\mathbf{u_4} = \lambda_1 \mathbf{u_1} + \dots \lambda_3 \mathbf{u_3}$, with $\lambda_i \neq 0$. Now use these terms as columns for the matrix

$$A = \begin{pmatrix} \lambda_1 \mathbf{u_1} & \lambda_2 \mathbf{u_2} & \lambda_3 \mathbf{u_3} \end{pmatrix}$$

Then it is easy to see that $t_A(P_i) = [\lambda_1 \mathbf{u_i}] = [\mathbf{u_i}]$ for $i < 4$ and $t_A(1 : 1 : 1) = [\lambda_1 \mathbf{u_1} + \dots \lambda_3 \mathbf{u_3}] = [\mathbf{u_4}]$.

Question 5.2. Prove that if $Q_1, \dots, Q_4$ and $R_1, \dots, R_4$ are two quadrangles then there is a unique projective transformation that maps Q_i to R_i.

ANSWER 5.2

5.1.3 Perspectivities

A geometric way to construct projective transformations comes from perspective drawings. In the previous lesson we saw that we can project one affine plane to another. Consider two planes π_1, π_2 in 3d-dimensional space not through the origin and a point $\mathbf{p} \neq 0$ not on those planes. Now define the following map: if U is a projective point we write it as $[\mathbf{u}]$ where $\mathbf{u} \in \pi_1$ lies in the first plane. Then project it from $\mathbf{p}$ onto the second plane. In other words let $\mathbf{v}$ be the intersection point between π_2 and the line through $\mathbf{p}$ and $\mathbf{u}$. The transformation $\mathbb{RP}^2 \to \mathbb{RP}^2 : [\mathbf{u}] \mapsto [\mathbf{v}]$ is called a *perspectivity*.

Question 5.3. Let $\pi_1 : z = 1$, $\pi_2 : y = 1$ and $\mathbf{p} = (1, 0, 0)$ and consider the perspectivity φ determined by these data. Find a matrix A such that $\varphi = t_A$.

ANSWER 5.3

Not every projective transformation is a perspectivity. A perspectivity always fixes the point $[\mathbf{p}]$ and all points on the intersection line $\{[\mathbf{u}] | \mathbf{u} \in \pi_1 \cap \pi_2\}$. For instance a rotation in the Euclidean plane is not a perspectivity because it only fixes one point.

However, just like all Euclidean transformations can be made by composing reflections, one can show that every projective transformation can be seen as the composition of at most three perspectives (See Brannan p171).

5.2 Projective invariance

Projective geometry studies concepts that are invariant under projective transformations. There are not that many Euclidean concepts that remain invariant under projective transformations because these maps do not preserve distances, angles or even parallelism. One of the few things that is preserved is the notion of a straight line and collinearity. This means that we can still speak of triangles and quadrangles.

Two figures in the projective plane are *projective congruent* if they can be mapped onto each other by a projective transformation. Because a projective transformation can map any quadrangle to any other, all quadrangles are projective congruent. This means that in projective geometry there is no difference between trapezoids, parallelograms, or other quadrangles.

5.2.1 The cross ratio

For four points on a line the situation is different. Let $P_1 = [\mathbf{u}_1], \dots, P_4 = [\mathbf{u}_4]$ be four points on a line. This means that there are $\mu_1, \mu_2 \nu_1, \nu_2 \in \mathbb{R}$ such that

$$\mathbf{u_3} = \mu_1 \mathbf{u_1} + \mu_2 \mathbf{u_2}$$
$$\mathbf{u_4} = \nu_1 \mathbf{u_1} + \nu_2 \mathbf{u_2}$$

These numbers depend on the vectors we have taken to represent the P_i. F.i. if we take $\lambda_1 \mathbf{u_1}$ instead of $\mathbf{u_1}$ then μ_1 changes to μ_1/λ_1. The ratio

$$(P_1, P_2; P_3, P_4) := \frac{\mu_2}{\mu_1} / \frac{\nu_2}{\nu_1}$$

is called the cross ratio of the four points.

Question 5.4. Show that the cross ratio only depends on the P_i not on the vectors representing them.

ANSWER 5.4

The cross ratio is in fact a projective invariant. If t_A is a projective transformation then $t_A(p_i)$ is represented by $A\mathbf{u}_i$ so we still have that

$$A\mathbf{u_3} = \mu_1 A\mathbf{u_1} + \mu_2 A\mathbf{u_2}$$
$$A\mathbf{u_4} = \nu_1 A\mathbf{u_1} + \nu_2 A\mathbf{u_2}$$

and therefore

$$(t_A P_1, t_A P_2; t_A P_3, t_A P_4) = (P_1, P_2; P_3, P_4).$$

Question 5.5. If P_1, P_2, P_3, P_4 are 4 points on a line with cross ratio $r = (P_1, P_2; P_3, P_4)$, show that there is a projective transformation that maps these to $(0:0:1)$, $(1:0:0)$, $(r:0:1)$ and $(1:0:1)$. Conclude that two quadruples of points on a line are projective congruent if and only if their cross ratios are the same.

ANSWER 5.5

The cross ratio of for points in the Euclidean plane can also be calculated using lengths. Suppose $P_1, \ldots, P_4 \in \mathbb{E}^2 \subset \mathbb{RP}^2$ are four points on a line. Consider this line as the x-axis of a coordinate system then we can identify these four points with real numbers $z_1, \ldots, z_4$. In homogeneous coordinates we get that $P_i = (z_i : 0 : 1)$. This enables us to calculate the μ_i, ν_i.

Question 5.6. Show that

$$(P_1, P_2; P_3, P_4) = \frac{z_3 - z_1}{z_3 - z_2} / \frac{z_4 - z_1}{z_4 - z_2}.$$

ANSWER 5.6

Now if we fix $z_1 = 0$, $z_4 = 1$, and let z_2 go to infinity, then the cross ratio becomes z_3. This means that the cross ratio gives the coordinate of P_3 in an affine coordinate system where P_1 is the zero P_4 is the one and P_2 lies at infinity. This is useful to determine distances on a perspective drawing.

Suppose that you have a picture of three points P, Q, R on a line drawn in perspective. If you know the real distance between P, Q and you want to calculate the real distance between P, R then you can extend the line to the horizon on the picture and call the intersection point S. We then have that

$$|PR| = (P, S; R, Q)|PQ|.$$

5.2.2 Conic sections

Let us now have look at conic sections in projective geometry. Remember that in affine geometry we could describe any conic section by a quadratic equation in (x, y). If we go to the projective plane this becomes a homogeneous quadratic equation which we can write as

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} A & C/2 & D/2 \\ C/2 & B & E/2 \\ D/2 & E/2 & F \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

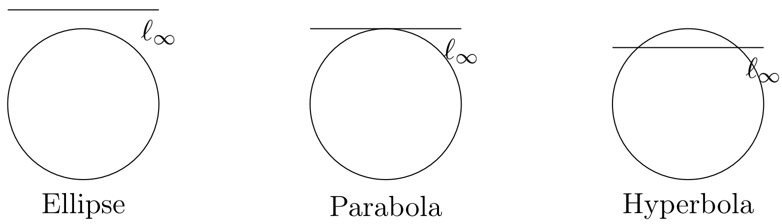
Let M be the symmetric 3×3-matrix in the equation. Under a projective transformation $\mathbf{x} \mapsto A\mathbf{x}$ this equation becomes

$$\mathbf{x}^\top M \mathbf{x} = 0 \rightsquigarrow \mathbf{x}^\top \underbrace{(A^{-1\top} M A^{-1})}_{\text{new } M} \mathbf{x} = 0.$$

If A is an orthogonal matrix then M becomes AMA^{-1}. Because every symmetric matrix can be diagonalized using orthogonal matrices, we can assume that $M = \mathrm{Diag}(\mu_1, \mu_2, \mu_3)$. If $\mu_1 \neq 0$ we can use a rescaling with $A = \mathrm{Diag}(|\mu_1|^{\frac{1}{2}}, 1, 1)$ to turn μ_1 into ± 1. Therefore we can assume that $\mu_i = 0, \pm 1$. By renaming the coordinates and possibly multiplying the equation with -1 we end up with the following possiblilities.

- $x^2 = 0$, which defines the projective line $x = 0$.
- $x^2 + y^2 = 0$, whose only solution is the projective point $(0 : 0 : 1)$.
- $x^2 - y^2 = 0$, which defines two projective lines $x = y$ and $x = -y$.
- $x^2 + y^2 + z^2 = 0$, which has no solutions.
- $x^2 + y^2 - z^2 = 0$, which is the projective equation for the unit circle.

The last possibility includes all equations coming from affine nondegenerate conic sections. Therefore we can conclude that projectively all nondegenerate conic sections are projective congruent. The only difference between ellipses, hyperbola and parabola is how they intersect the line at infinity and this depends on how the affine plane is embedded in $\mathbb{RP}^2$.



This is interesting because it means that to prove a projective theorem about a conic section we can choose special coordinates and assume it is a circle. Take the following statement:

If a conic section is tangent to the three sides of a triangle then the three lines connecting the vertices of the triangle with the tangent points of the opposite sides are concurrent.

Using a projective transformation we can assume that the conic section is the unit circle and therefore it suffices to prove the statement for a circle.

5.3 Duality

Apart from the projective transformations, projective geometry possesses another kind of symmetry: one between points and lines. In the projective plane there exists one line through every pair of points but also there is precisely one point that lies on every pair of lines.

$$\forall P_1, P_2 : \exists \ell : P_1 \in \ell \text{ and } P_2 \in \ell$$
$$\forall \ell_1, \ell_2 : \exists P : \ell_1 \ni P \text{ and } \ell_2 \ni P$$

If we swap the roles of points and lines and reverse the inclusion these two statements are transformed into each other.

Similarly every statement about points and lines in the projective plane can be transformed in a dual statement where the roles of points and lines are swapped. For instance Pappus's theorem

Consider two lines ℓ_1, ℓ_2 and six points $A, B, C \in \ell_1$, $A', B', C' \in \ell_2$. Let D, E, F be the intersection point between BC' and CB', AC' and CA', and AB' and $A'B$. Then D, E, F lie on a line.

becomes

Consider two points P_1, P_2 and six lines $a, b, c \ni P_1$, $a', b', c' \in P_2$. Let d, e, f be the lines through $b \cap c'$ and $c \cap b'$, $a \cap c'$ and $c \cap a'$, and AB' and $A'B$. Then d, e, f go through one point.

Question 5.7. Draw a picture of the second theorem.

ANSWER 5.7

These theorems are called *dual theorems.* If you have a proof for one of the theorems you can turn it into a proof of the second. Note that every line in $\mathbb{RP}^2$ is represented by an equation $ax+by+cz=0$ where a, b, c are not all zero. If we multiply these coefficients with a scalar we get the same line so we can associate to each line a projective point and vice versa

$$\ell : ax + by + cz = 0 \mapsto \rho(\ell) := (a : b : c)$$
$$P = (u : v : w) \mapsto \rho(P) : ux + vy + wz = 0$$

This procedure is called a polarity. It switches the roles of lines and points, and if P lies on ℓ then $\rho(\ell)$ lies on $\rho(P)$. Therefore if you apply ρ to all objects that appear in the proof of a theorem you get a proof for the dual theorem.

Question 5.8. Show that if P is point on the unit circle $x^2 + y^2 - z^2 = 0$ then $\rho(P)$ is a line tangent to the unit circle.

ANSWER 5.8

More generally one can show that if P lies on the conic section with equation $\mathbf{x}^\top K\mathbf{x} = 0$ then $\rho(P)$ is tangent to the dual conic section with equation $\mathbf{x}^\top K^{-1}\mathbf{x} = 0$. Therefore it is also possible to dualize statements about conic sections, if we translate *lying on* to is *tangent to.*

5.4 A notch more abstract

5.4.1 Projective space

It is possible to associate a projective space to each real vector space V. The points of this space correspond to the 1-dimensional subspaces of V

$$\mathbb{P}(V) = \{\mathbb{R}v \mid v \in V \setminus \{0\}\}$$

If $v \in V$ is a nonzero vector, then we denote the corresponding projective point by $[v] = \mathbb{R}v$.

If $V = \mathbb{R}^3$ then we get the projective plane. Notice that the plane is 2-dimensional while V is 3-dimensional. Similarly if $V = \mathbb{R}^{k+1}$, we call $\mathbb{P}(V)$ the real k-dimensional projective space and also denote it by $\mathbb{RP}^k$.

If $W \subset V$ is an $r+1$-dimensional subspace then we also have that $\mathbb{P}(W) \subset \mathbb{P}(V)$, so $\mathbb{P}(W)$ is a projective subspace of $\mathbb{P}(V)$. If we return to the case $V = \mathbb{R}^3$ and $W = \{(x, y, z) \in V \mid x = y\}$ then $\mathbb{P}(W)$ can be seen as the projective line in $\mathbb{P}(V) = \mathbb{RP}^2$ given by the equation $X - Y = 0$.

Analogously, the three-dimensional vector space $W = \{(x, y, z, w) \in \mathbb{R}^4 \mid x + y + z + w = 0\}$ gives rise to a projective plane $\mathbb{P}(W)$ sitting inside the projective 3-dimensional space $\mathbb{RP}^3 = \mathbb{P}(\mathbb{R}^4)$. The two-dimensional vector space $U = \{(x, y, -x, -y) \mid x, y \in \mathbb{R}\}$ gives a projective line $\mathbb{P}(U) \in \mathbb{RP}^3$ that lies in the plane $\mathbb{P}(W)$.

5.4.2 Projective transformations

For a vector space V the general linear group is the group of linear transformations from V to itself.

$$\mathrm{GL}(V) = \{\psi : V \to V \mid \psi \text{ is linear}\}$$

If $V = \mathbb{R}^{k+1}$ we can identify this group with the group of invertible $(k+1) \times (k+1)$-matrices. These linear transformations map a one-dimensional subspace to a one-dimensional subspace, so $GL(V)$ also acts on the projective space $\mathbb{P}(V)$. A scalar matrix $\lambda \cdot \mathrm{id}_V$ maps every point in $\mathbb{P}(V)$ to itself, so it acts trivially. Therefore we define the group of projective linear transformations as

$$\mathrm{PGL}(V) = \mathrm{GL}(V)/(\mathbb{R}^* \cdot \mathrm{id}_V).$$

This is the symmetry group of the projective space $\mathbb{P}(V)$.

5.4.3 Duality

If V is a $k+1$-dimensional vector space, we can define its dual vector space as the space of all linear maps from V to $\mathbb{R}$:

$$V^\vee = \{\varphi : V \to \mathbb{R} \mid \varphi \text{ is a linear map}\}.$$

You can easily check that this is a vector space if we put

$$(\varphi_1 + \varphi_2)(\mathbf{v}) := \varphi_1(\mathbf{v}) + \varphi_2(\mathbf{v})$$
$$(\lambda\varphi)(\mathbf{v}) := \lambda(\varphi(\mathbf{v})).$$

This vector space is again $k+1$-dimensional and if we identify $V = \mathbb{R}^{k+1}$ with the space of column vectors, then we can identify $V^\vee$ with the space of row vectors where $b = \begin{pmatrix} b_0 & \cdots & b_k \end{pmatrix}$ corresponds to the map

$$v \mapsto bv = b_0 v_0 + \ldots + b_k v_k$$

the elements of $V^\vee$ are sometimes called covectors.

Each nonzero covector $\varphi \in V^\vee$ defines a subspace $W = \mathrm{Ker}\ \varphi = \{v \in V \mid \varphi(v) = 0\} \subset W$ and the projective space $\mathbb{P}(W)$ describes a $k-1$-dimensional projective subspace of $\mathbb{P}(W)$. Such a space is called a hyperplane. In particular in $\mathbb{RP}^2$ the hyperplanes are projective lines, in $\mathbb{RP}^3$ they are projective planes.

Note that φ and $\lambda\varphi$ have the same kernel, so they define the same hyperplane. Therefore the set of hyperplanes in $\mathbb{P}(V)$ is in fact in one-to-one correspondence with the set of points in $\mathbb{P}(V^\vee)$. The latter is called the dual projective space.

Vice versa the hyperplanes in $\mathbb{P}(V^\vee)$ correspond to the points in $\mathbb{P}(V)$ by associating to each point $p = [v]$ the hyperplane $P = \mathbb{P}(\mathrm{Ann}_v)$ with

$$\mathrm{Ann}_v = \{\varphi \in V^\vee \mid \varphi(v) = 0\}.$$

The points in $P = \mathbb{P}(\mathrm{Ann}_v)$ are the hyperplanes in $\mathbb{P}(V)$ that go through p. So the roles of points and hyperplanes are reversed if we go to the dual picture.

Question 5.9. Show that if the point $p \in \mathbb{P}(V)$ lies on the hyperplane $H \subset \mathbb{P}(V)$ then dual point $h \in \mathbb{P}(V^\vee)$ associated to that hyperplane lies on the dual hyperplane $P \subset \mathbb{P}(V^\vee)$ coming from p.

ANSWER 5.9

Remark 5.10. Instead of working with real vector spaces we can also work with complex vector spaces. In this case the set of one-dimensional complex subspaces of $\mathbb{C}^{k+1}$ is called the k-dimensional complex projective space and denoted by $\mathbb{CP}^k$. Complex algebraic geometry studies subsets of this space defined by homogeneous equations and is an important topic in the Master.

5.5 Further practice

Exercise 5.1. [Exercise 3.3.1 from Brannan, p. 192-193] Determine which of the following transformations $t : \mathbb{RP}^2 \to \mathbb{RP}^2$ are projective:

(a) $t([X : Y : Z]) := [2X : Y + 3Z : 1]$,

(b) $t([X : Y : Z]) := [X : X - Y + 3Z : X + Y]$,

(c) $t([X : Y : Z]) := [2Y : Y - 4Z : X]$,

(d) $t([X : Y : Z]) := [X + Y - Z : Y + 3Z : X + 2Y + 2Z]$.

Exercise 5.2. [Problem 3.3.4 from Brannan, p. 193] Let t be the projective transformation given by $t([X : Y : Z]) = [2X + Y : -X + Z : Y + Z]$. Find the image of the line given by the equation $X + 2Y + 3Z = 0$ under t.

Exercise 5.3. [Exercise 3.3.5 from Brannan, p. 193] Determine a matrix that maps the Points $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$ and $[1 : 1 : 1]$ on the Points

(a) $[-2 : 0 : 1], [0 : 1 : -1], [-1 : 2 : -1]$ and $[-1 : 1 : -1]$;

(b) $[0 : 1 : 0], [1 : 0 : 0], [-1 : -1 : 1]$ and $[2 : 1 : 1]$;

(c) $[0 : 1 : -3], [1 : 1 : -1], [4 : 2 : 3]$ and $[7 : 4 : 3]$ respectively.

Exercise 5.4. [Exercise 3.3.6 (a) from Brannan, p. 193] Determine the projective transformation that maps the points of the previous exercise (a) onto the points of (b).

Exercise 5.5. [Problem 3 from Brannan, p. 178] Prove the dual of Pappus theorem. Hint chose a coordinate system with $P_1 = (1 : 0 : 0), P_2 = (0 : 1 : 0)$ and $b \cap c' = \{(0 : 0 : 1)\}$ and $c \cap b' = \{(1 : 1 : 1)\}$.

Exercise 5.6. [Exercise 3.4.2 from Brannan, p. 194] Let $\triangle ABC$ be a triangle in $\mathbb{R}^2$, and take $U \in \mathbb{R}^2$ such that U is not on one side of $\triangle ABC$ lies. Let P be the intersection of (the extensions of) BC and AU, Q be the intersection of (the extensions of) CA and BU and R be the intersection of (the extensions of) AB and CU. Show that P, Q and R are not collinear.

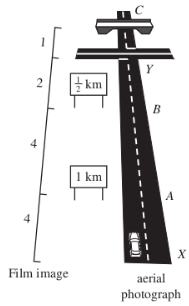
Exercise 5.7. [Problem 3.5.1 from Brannan, p. 194] For each of the following sets of Points A, B, C, D, calculate the cross ratio $(ABCD)$.

(a) $A = [2 : 1 : 3], B = [1 : 2 : 3], C = [8 : 1 : 9], D = [4 : -1 : 3]$,

(b) $A = [2 : 1 : 1], B = [-1 : 1 : -1], C = [1 : 2 : 0], D = [-1 : 4 : -2]$,

(c) $A = [-1 : 1 : -1], B = [0 : 0 : 2], C = [5 : -5 : 3], D = [-3 : 3 : 7]$.

Exercise 5.8. [Exercise 3.5.5 from Brannan p. 194-195] The diagram represents an aerial photograph of a straight road on flat ground. At A there is a sign 'Junction 1 km', at B a sign 'Junction 1/2 km', and C is the road junction. Also, a police patrol car is at X , and a bridge is at Y . The distances marked on the left of the diagram are measured in cm from the photograph. Calculate the actual distances (in km) of the patrol car and the bridge from the junction.



Exercise 5.9. [Exercise 4.3.6 (a) - (d) from Brannan p. 254] The Points $P = [1 : -1 : 1], Q = [1 : -2 : 2], R = [1 : -2 : 1]$ lie on the projective conic E with equation

$$-2X^2 + 3XY + 3Y^2 + 6XZ + 6YZ + 2Z^2 = 0.$$

(a) Verify that the projective transformation given by

$$t([\mathbf{x}]) := [A\mathbf{x}]$$

with

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

maps P, Q and R to $[1:0:0], [0:1:0]$ and $[0:0:1]$ respectively.

(b) Check that

$$A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -2 & 2 \\ 1 & 2 & -1 \end{pmatrix}.$$

(c) Find the equation for the projective conic $t(E)$.

(d) Find a projective transformation that maps E to the projective conic with equation

$$XY + YZ + ZX = 0.$$

Exercise 5.10. [Exercise 4.4.1 from Brannan p. 254-255] Let E be the projective conic with equation

$$X^2 + 2XY + 3Y^2 + 6XZ + 2YZ + Z^2 = 0.$$

(a) Find a matrix K associated with E.

(b) Give an orthogonal matrix P such that $P^T KP$ is a diagonal matrix.

(c) Give a projective transformation that maps E on the projective conic with equation $X^2 + Y^2 = Z^2$.

Exercise 5.11. Let ρ be the polarity that maps a point $P = (u : v : w)$ to the line $\ell : ux+vy+wz = 0$ and vice versa.

(a) Show that if t_A is a projective transformation then $\rho(t_A(P)) = t_{(A^{-1})^T}\rho(P)$.

(b) Show that if ℓ is tangent to the nondegenerate conic section E with equation $\mathbf{x}^\top K\mathbf{x} = 0$ then $\rho(\ell)$ is a point on the conic section with equation $\mathbf{x}^\top K^{-1}\mathbf{x} = 0$. (Hint: use a projective transformation that sends E to the unit circle $x^2 + y^2 - z^2 = 0$ and use the first part of the exercise).

Instageom 5. An Anamorphosis is a distorted projection or perspective requiring the viewer to occupy a specific vantage point, use special devices or both to view a recognizable image. Below is an example of this.

In the picture the ambassadors the painter, Holbein, has painted a skull in perspective. This skull is best seen if you view the picture from an angle. Try to make an anamorphosis yourself.

Chapter 6

Hyperbolic geometry

6.1 Origins

Ever since the Elements was published, mathematicians tried to prove Euclid' fifth postulate from the others because it looked less obvious and more complicated then the rest. Many claimed they succeeded but time and again these proofs turned out to be faulty because they only showed that the postulate implied a different statement which seemed at first sight absurd, but which could also not be disproved using Euclid's other postulates.

Question 6.1. Look up the notion of a Saccheri quadrilateral and explain how Saccheri used this notion in his quest to prove the fifth postulate by considering three cases.

ANSWER 6.1

By the beginning of the 19th century trying to prove the fifth postulate was considered a bad career move for a mathematician and most mathematicians were working on other more interesting things. In his private studies the young genius Carl Friedrich Gauss, investigated what happened when you changed the Playfair's axiom such that there are at least two lines through a point that do not intersect a given line. He came to the conclusion that you could develop a new strange geometry which differed from Euclidean geometry but was internally consistent: it was not possible to find contradictions. Gauss did not published these results so they were not known to other mathematicians.

Some years later, the Hungarian Janos Bolyai and the Russian Nicolai Lobachevski did more or less the same thing as Gauss. Both published their results in obscure places and they remained forgotten until they were rediscovered around 1860. It seemed that there were two different geometries Euclidean and a new one, *hyperbolic geometry*, which seemed equally consistent. But unlike the Euclidean plane which can be modeled as $\mathbb{R}^2$ using analytical geometry there was no model known for this new kind of geometry.

6.2 The Beltrami-Klein or Cayley-Klein Model

In this section we will construct the Beltrami-Klein or Cayley-Klein model for hyperbolic geometry. To construct this model, we first will have a look at how we can construct affine geometry from projective geometry.

Consider the projective plane $\mathbb{RP}^2$ and a line ℓ. Without loss of generality we can assume that ℓ has the equation $z = 0$. Now we look at the projective transformations that map ℓ to itself.

Question 6.2. Show that these transformations can be written as t_A where A is of the form

$$\begin{pmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ 0 & 0 & 1 \end{pmatrix}$$

conclude from this that

$$\{t_A \in \mathrm{PGL}(3, \mathbb{R}) \mid t_A(\ell) = \ell\} \cong \mathrm{Aff}_2$$

ANSWER 6.2

Therefore we can recover affine geometry as the space $\mathbb{RP}^2 \setminus \ell$ together with the subgroup of $\mathrm{PGL}(3, \mathbb{R})$ that maps ℓ to itself.

Now let us mimic this procedure for a conic section instead of a line. Let K be the unit circle $x^2 + y^2 = z^2$. This conic section divides $\mathbb{RP}^2 \setminus K$ in two parts

- The interior of K. These are points P for which every line through P intersects K in two points.
- The exterior of K. These are points P for which there is a line through P not intersecting K.

Every projective transformation that maps K to itself, maps points inside K to points inside K. Therefore it makes sense to define a new geometry: the hyperbolic plane $\mathbb{H}^2_{BK}$.

- the hyperbolic points are the points in the interior of K.
- the hyperbolic lines are intersections of projective lines with the interior of K.
- the hyperbolic symmetries are the projective transformations that map K to itself.

In this new geometry it is also possible to define a distance. If $P, Q \in \mathbb{H}^2_{BK}$ then we get two extra points for free: the intersection points S, T of the line PQ with K. These extra points can be used to define

$$d_{BK}(P, Q) = \frac{1}{2} |\ln |(S, T; P, Q)||$$

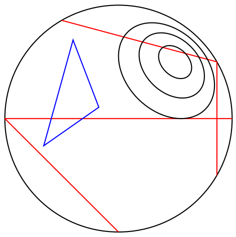
Note that if P or Q move towards the circle the cross ratio will become zero or infinity, which after taking the logarithm results in a distance approaching infinity.

Question 6.3. The reason why we take the logarithm is because we want distances to be additive. Show that if P, Q, R lie on a line such that Q lies between P and R then

$$d_{BK}(P, R) = d_{BK}(P, Q) + d_{BK}(Q, R)$$

ANSWER 6.3

Because hyperbolic symmetries are projective transformations they preserve the cross ratio and map the intersection points with the conic section to intersection points with the conic section. Therefore these hyperbolic transformations will preserve the hyperbolic distance, so they are isometries. The notion of distances allows us to define hyperbolic circles.

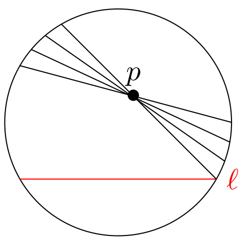


Some hyperbolic lines, circles and a hyperbolic triangle

The final thing we need in our new geometry is a way to measure angles. This is the hardest part in this model. Note that in general the projective transformations that preserve the circle do not preserve angles, so the ordinary notion of angles will not do. There is only one exception: the hyperbolic isometries that map the center of the circle $O = (0 : 0 : 1)$ to itself are just the rotations in the Euclidean plane around the origin which do preserve the normal angles. Therefore we can define hyperbolic angles as follows

- If the corner of the angle is $O = (0 : 0 : 1)$ we define the angle as the Euclidean angle.
- If the corner of the angle is P then first apply a hyperbolic isometry to map P to O and then measure the angle at O.

Now we have all the same notions as we have in Euclidean geometry: points, lines, circles, distances and angles. But how does Hyperbolic geometry compare to Euclidean geometry? You can check that all basic postulates that are true in Euclidean geometry (or more precisely all Hilbert's axioms) are also true in Hyperbolic geometry except one: Playfair's axiom. In hyperbolic geometry you can draw more than one line through a point that does not intersect a given line. There are two lines that meet the given line on the circle, these are called the parallels and the the lines that meet the given lines outside the circle are called the ultra-parallels.



A parallel and three ultraparallel lines to ℓ through p

The hyperbolic plane is similar to the Euclidean plane but in a subtle way.

- Some of its theorems are exactly the same. For example the fact that the medians of a triangle intersect in one point.
- Other theorems are almost the same. The sine rule in a triangle with sides a, b, c and angles α, β, γ becomes
$$\underbrace{\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}}_{\text{Euclidean}} \rightsquigarrow \underbrace{\frac{\sinh a}{\sin\alpha} = \frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\gamma}}_{\text{Hyperbolic}}$$
- Some theorems are weird or counter-intuitive. E.g. the area of a triangle cannot be bigger than π.

Question 6.4. The hyperbolic Pythagoras theorem states that if a, b, c are the sides of a right angled triangle and c is the hypotenuse then
$$\cosh c = \cosh a \cosh b.$$
Show that for very small triangles this theorem becomes in the limit the Euclidean Pythagorean theorem.

ANSWER 6.4

6.3 Other models

The Beltrami-Klein model is not the only model in town. There are two other models that have their own advantages.

6.3.1 The Poincaré disk model

In the *Poincaré disk model* $\mathbb{H}^2_{PD}$ the points are again the points inside the unit circle, but the lines are diameters and circle segments that stand perpendicular to the unit circle. The angles in this model are the ordinary Euclidean angles, but now the distances are more difficult to define. If we view $x, y \in \mathbb{H}^2_{PD}$ as complex numbers with norm $|x|, |y| < 1$ then
$$d_{PD}(x, y) := \cosh^{-1}\left(\frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)} + 1\right)$$
This model is also used in the paintings of Escher *Circle Limit I-IV*

Question 6.5. Look up the notion of a circle inversion and write down its definition. In the Poincaré disk model the circle inversions about hyperbolic lines play the role of reflections: they are isometries that fix all points on a line.

ANSWER 6.5

6.3.2 Poincaré half plane model

In the *Poincaré half plane model* $\mathbb{H}^2_{PH}$ the points are the complex numbers with positive imaginary part. The hyperbolic lines are vertical rays and half circles that stand perpendicular on the real axis. Again the angles are the Euclidean angles and the distance is

$$d_{PH}(x,y) := \cosh^{-1}\left(\frac{|x-y|^2}{2\cdot \operatorname{Im} x \cdot \operatorname{Im} y} + 1\right)$$

In this model the group of direct isometries can be written as

$$\mathrm{PSL}(2c,\mathbb{R}) := \{u : x \mapsto \frac{ax+b}{cx+d} \mid a,b,c,d \in \mathbb{R}, ad-bc = 1\}$$

6.3.3 Connections

Between these three models there are maps that allow us to switch from one model to another.

$$\begin{tikzcd} \mathbb{H}^2_{BK} \arrow[r, bend left, "x \mapsto \frac{x}{1+\sqrt{1-|x|^2}}"] & \mathbb{H}^2_{PD} \arrow[l, bend left, "x \mapsto \frac{2x}{1+|x|^2}"] \arrow[r, bend left, "x \mapsto \frac{x+i}{ix+1}"] & \mathbb{H}^2_{PH} \arrow[l, bend left, "x \mapsto \frac{x-i}{1-ix}"] \end{tikzcd}$$

Question 6.6. Show that the map from $\mathbb{H}^2_{BK}$ to $\mathbb{H}^2_{PD}$ maps lines (not through the origin) to circles.

ANSWER 6.6

6.4 Further Practice

Exercise 6.1. Calculate the distance between $(1/4, 0)$ and $(0, 1/2)$ in the Beltrami-Klein model.

Exercise 6.2. Find an expression for the distance in the Beltrami-Klein model in terms of the coordinates of two points.

Exercise 6.3. Show that the sum of the angles in a hyperbolic triangle is smaller than π. Use the Poincare disc model.

Exercise 6.4. Show that all hyperbolic circles are ellipses in the Beltrami-Klein model.

Exercise 6.5. Show that all hyperbolic circles are circles in the Poincaré disk model.

Instageom 6. Take one of the paintings of Escher and turn it from the Poincaré disk model into the Beltrami-Klein model. Make a hyperbolic tiling in one of the three models. Take a picture and apply a circle inversion to it.

Chapter 7

New Geometries

In this final chapter we will look at how geometry evolved and diversified in the last 150 years. In particular we will look at a couple of different visions on what geometry is according to some famous mathematicians and tie this to courses on algebra and geometry that are given in the bachelor and master mathematics.

7.1 Riemann's Habilitations speech

In 1854 Bernard Riemann had to give a speech in order to become professor at the University of Göttingen. He had proposed three titles out of which Gauss, also a professor at Göttingen, chose *On the hypotheses that lie at the foundations of Geometry.* In his talk Riemann defined a new kind of space: the Riemannian manifold.

A *differentiable manifold* consists of a set of points that can locally be described using a set of n coordinates. The number n is the dimension of the manifold and using these coordinates one can do analysis on the manifold (differentiation and integration). A *Riemannian manifold* is a differentable manifold with a metric. That is a way to measure lengths of trajectories in the space. One good example to keep in mind is the sphere. Locally you can describe it using two coordinates and it is possible to calculate the length of a curve on the sphere. Other examples are the Euclidean plane and the hyperbolic plane.

With these two ingredients it is possible to define a lot of concepts that are important in geometry: straight lines (these are geodesics: trajectories with minimal length), distances (the length of the shortest trajectory between two points) and angles (using local trigonometry), areas and volumes (using integrals), etc.

Inspired by his mentor Gauss, Riemann also introduced a notion of curvature in these spaces. The total curvature of a geodesic triangle is the sum of its angles minus π. So in Euclidean geometry all angles are flat but in hyperbolic geometry all triangles have negative curvature.

Question 7.1. On a sphere with radius R the geodesics are the big circles, these are circles with center the center of the sphere and radius R. Now consider a triangle made of the equator and two meridians that make an angle θ. What is the total curvature of that triangle and what is its area?

ANSWER 7.1

The sectional curvature of a point in a Riemannian manifold in two directions is the ratio between the total curvature of a small geodesic triangle with one corner in that point and two sides in those directions. The curvature of a space can differ in different points and for different directions. A space of constant curvature is a space for which in all points and in all directions is the same. There are three important 2-dimensional spaces with constant curvature.

- a sphere with radius 1 has constant curvature $+1$,
- the Euclidean plane has constant curvature 0,
- the hyperbolic plane has constant curvature -1.

Riemann generalized these three spaces to arbitrary dimensions and speculated that it might be important to study the curvature of our own space. More than fifty years later Einstein used a 4-dimensional version of Riemannian geometry to describe gravity as curvature in spacetime.

If you are interested in learning more about manifolds, you can follow the course *Differential Geometry* `51228DIF6Y` in the third year.

7.2 The Erlangen Program

When Felix Klein became a professor of Mathematics at the University of Erlangen in 1872 he had to outline a research program. In this program Klein tried to find a unifying principle behind all the new geometries that had arisen in the 19th century (projective geometry , hyperbolic geometry, inversion geometry, higher-dimensional geometry, Riemannian geometry,...). He came up with the idea of using another newly discovered mathematical concept for this purpose: groups.

For Klein a geometry consisted of two things: a set of points and a group of symmetries acting on it. All the rest followed out of this pair. Doing geometry from this point of view is to look at configurations of points up to these symmetries: two configurations are congruent if there is a symmetry that maps one configuration onto the other. Important properties in this geometry are properties that remain invariant under the symmetries. The theorems of the geometry describe relations between the invariant properties.

The second idea of Klein was that you can create new geometries from old ones by allowing more or less symmetries. We have seen this principle throughout the course.

- We turned Euclidean geometry into affine geometry by enlarging the group of Euclidean transformations to the group of affine transformations.
- We turned projective geometry into hyperbolic geometry be restricting the group of projective transformations to the subgroup that fixes the unit circle.

Finally, Klein showed that projective geometry was the mother of almost all the geometries that were known in his time. He gave constructions for most of these, starting from projective geometry and restricting to certain subgroups of projective transformations. He also speculated that it would be possible to enlarge the group of projective transformations to even larger groups, such as all algebraic, differentiable or continuous transformations. These geometries are now known as *algebraic geometry*, *differential geometry* and *topology*. The latter two are taught in the Bachelor of mathematics, while algebraic geometry is an important subject in the master `53341ALG8Y`, `53342ALG8Y`.

Question 7.2. How would you describe spherical geometry using Klein's Erlangen program (what is the space, and what is the group). Could you also make spherical geometry out of projective geometry?

ANSWER 7.2

Klein's vision of geometry as a space with a group of symmetries begs the question. Is it possible to classify groups of symmetry and the spaces on which they act? This was studied by a good friend of Klein, the Norwegian Sophus Lie. Groups of symmetries like the group $\mathrm{PGL}(3, \mathbb{R})$ or $\mathrm{Aff}(2, \mathbb{R})$ are now known as *Lie groups* and there is a whole theory on how to classify and describe these. How groups act on spaces is known as representation theory. Both Lie groups and representation theory are very important areas in mathematics and to know more about them you can follow *Representation theory* `5122REPR6Y` in the third year and *Lie Groups* `53348LIG8Y` in the master.

7.3 Topology

In 1895 Poincaré published a manuscript in which he laid the foundations of another kind of geometry, which he called Analysis Situs, but which nowadays is better known as (algebraic) topology. His idea was to develop new methods that could describe aspects of spaces that are preserved under continuous transformations. The classical example of such continuous transformations is turning a cup into a doughnut.

Both spaces have one hole, but you do you describe that hole mathematically? Poincaré came up with two methods: simplicial homology and the fundamental group. For the first method you divide the surface in small triangles and count

$$\#\{\text{vertices}\} - \#\{\text{edges}\} + \#\{\text{triangles}\}$$

This number is called the Euler charateristic and it does not depend on the particular triangles, only on the surface up to continuous transformations.

Question 7.3. Calculate this number for a couple of examples where the surface is a sphere or a doughnut. In the former case the answer is always two, in the latter it is always zero.

ANSWER 7.3

If you want to know more about the second method (the fundamental group) you have to follow the course *Topology* `5122TOPO6Y` in the second semester of the second year. Homology is an important part of *Algebraic topology*, which is taught in the master `53341ALT8Y`, `53342ALT8Y`.

7.4 Categories

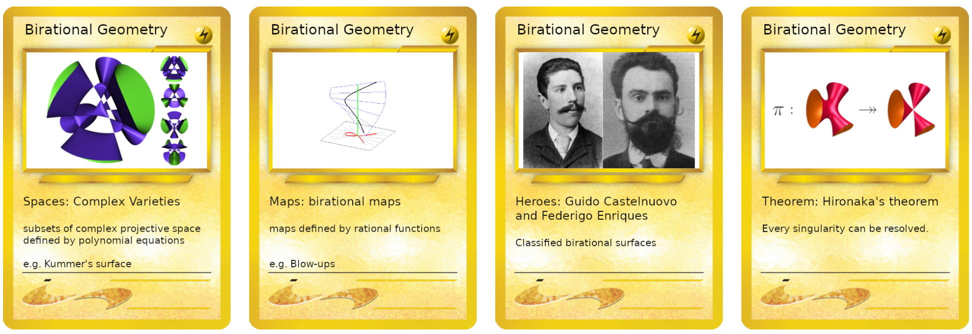
Klein envisioned geometry as the study of one space with a set of symmetries, but in the twentieth century it became clear that it is important to look at a lot of spaces of the same type together and explore the relations between them. There are for instance a lot of different topological spaces (e.g. a sphere, a torus, a circle) and continuous maps between them. We can describe them all together using categories.

A *category* consists of objects and morphisms between them and for almost every mathematical concept there is a category associated to it. In particular to every type of algebra or geometry we can associate a category

Geometry	Objects	Morphisms
Topology	Topological spaces	Continuous maps
Differential geometry	Manifolds	Smooth maps
Algebraic geometry	Varieties	Algebraic maps
Lie theory	Lie groups	Lie group morphisms
Representation theory	Modules	Module morphisms

Categories were introduced in 1945 by Saunders Mac Lane and Samuel Eilenberg, two friends who had worked together during the war. They form an abstract framework to describe connections between different types of geometry and algebra and it has become the language to study a lot of aspects of topology and modern algebraic and differential geometry. If you are interested in this, you can follow the course *Modules and categories* `5122MOCA6Y` in the third year.

Instageom 7. Make a geometry quartets. Choose a geometry from `https://en.wikipedia.org/wiki/List_of_geometry_topics` and make four cards. One with the type of spaces that are studied, one with the type of maps, one with a famous geometer from that branch, and one with a famous theorem.



Appendix

Below is a brief dictionary of some non-obvious geometric terms.

English	**Nederlands**
adjacent	aanliggend
altitude	hoogtelijn
angle (acute, right, obtuse, straight)	hoek (scherp, recht, stomp, gestrekt)
angle bisector	bissectrice
arc	boog
centre	middelpunt
chord	koorde
circumscribed	omgeschreven
concurrent	concurrent, samenkomend
conic (section)	kegelsnede
cross-ratio	dubbelverhouding
curvature	kromming
cyclic quadrilateral	koordenvierhoek
degenerate	ontaard
directrix	richtlijn
eccentricity	excentriciteit
equilateral	gelijkzijdig
focus	brandpunt
hull	omhulsel
inscribed	ingeschreven
isosceles	gelijkbenig
manifold	gladde variëteit
median	zwaartelijn
opposite	overstaand
parallel	evenwijdig
paralle***l***ogram	paralle***ll***ogram
perpendicular	loodrecht
perpendicular bisector	middelloodlijn
quadrangle, quadrilateral	vierhoek
radius	straal
rhombus	ruit
right-angled	rechthoekig
tangent line	raaklijn
trapezium, trapezoid	trapezium