4
Quivers

Representations of quivers have been studied intensively and there are many nice references such as [GR92, Sch14, CB99, CB92, Bri12]. The aim of this chapter is to view these classical results from the perspective of A_∞-categories. We will work out a couple of examples for which we can classify all representations. Each of these examples can be interpreted geometrically in two different ways. This will give us a first glimpse of homological mirror symmetry.

4.1 Representations of Quivers

4.1.1 The Path Algebra

Definition 4.1 A *quiver* Q is an oriented graph. We denote the set of vertices by Q_0, the set of arrows by Q_1 and the maps $h, t\colon Q_1 \to Q_0$ assign to each arrow its head and tail.

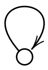
A *nontrivial path* p of length k is a sequence of arrows $a_1 \cdots a_k$ such that $t(a_i) = h(a_{i+1})$. A *trivial path* is just a vertex $v \in Q_0$. A path $a_1 \cdots a_k$ is called *cyclic* if $h(a_1) = t(a_k)$ and the equivalence class of a cyclic path under cyclic permutation is called an *oriented cycle*.

Definition 4.2 The *path algebra* $\mathbb{C}Q$ is a vector space spanned by all paths. The product of two paths is their concatenation if possible and 0 otherwise. Paths are concatenated from right to left: $pq = \circ \xleftarrow{p} \circ \xleftarrow{q} \circ$.

The trivial paths corresponding to the vertices v are a complete set of orthogonal idempotents:

$$\forall\, v, w \in Q_0 \text{ we have } vw = \begin{cases} v, & v = w, \\ 0, & v \neq w \end{cases} \quad \text{and} \quad \sum_{v \in Q_0} v = 1.$$

They span a semisimple subalgebra, which we will denote by $\mathbb{C}Q_0$. All nontrivial paths span a two-sided ideal $J = \langle a \mid a \in Q_1 \rangle \lhd \mathbb{C}Q$.

Example 4.3 If we take for example the quivers $\circ$,  and $\circ \longleftarrow \circ$ then the corresponding path algebras are $\mathbb{C}$, $\mathbb{C}[X]$ and the nonstrictly upper-triangular (2×2)-matrices $\left(\begin{smallmatrix}\mathbb{C} & \mathbb{C}\\ 0 & \mathbb{C}\end{smallmatrix}\right)$. A quiver with one vertex and k loops gives rise to the free algebra $\mathbb{C}\langle X_1, \ldots, X_k\rangle$.

Remark 4.4 The path algebra can also be considered as a category $\mathbb{C}\mathtt{Q}$. The objects are the vertices, the hom-space between two vertices is $\mathbb{C}\mathtt{Q}(v, w) = w\mathbb{C}Qv$ and the vertex idempotents are the identities $\mathbb{1}_v = v \in v\mathbb{C}Qv$.

4.1.2 Simples and Projectives

We will denote the categories of finite-dimensional, finitely generated and all right $\mathbb{C}Q$-modules by $\mathtt{mod}\,Q$, $\mathtt{Mod}\,Q$ and $\mathtt{MOD}\,Q$ respectively and their A_∞-versions by $\mathtt{mod}^\bullet\,Q$, $\mathtt{Mod}^\bullet\,Q$ and $\mathtt{MOD}^\bullet\,Q$. There are two important sets of right modules parametrized by Q_0:

- the *basic projectives* $P_v := v\mathbb{C}Q$, which can also be viewed as contravariant functors $\mathcal{P}_v \colon \mathbb{C}\mathtt{Q} \to \mathtt{VECT}(\mathbb{C})$ with $\mathcal{P}_v(w) = v\mathbb{C}Qw$ and $\mathcal{P}_v(a)x = xa$;
- the *basic simples* $S_v := v\mathbb{C}Q/J$, which can also be viewed as contravariant functors $\mathcal{S}_v \colon \mathbb{C}\mathtt{Q} \to \mathtt{VECT}(\mathbb{C})$ with $\mathcal{S}_v(w) = \delta_{vw}\mathbb{C}$ and $\mathcal{S}_v(a) = 0$.

Because the former are projective there are no nontrivial extensions between them so

$$\mathrm{Ext}^i_{\mathbb{C}Q}(P_v, P_w) = \begin{cases} w\mathbb{C}Qv, & i = 0,\\ 0, & i > 0.\end{cases}$$

This implies that the full subcategory $\mathtt{P}^\bullet \subset \mathtt{D}\,\mathtt{MOD}^\bullet\,Q$ containing the basic projectives is isomorphic to $\mathbb{C}\mathtt{Q}$ viewed as an A_∞-category concentrated in degree 0 with trivial higher products.

The simple module S_v has a projective resolution

$$\bigoplus_{h(a)=v} P_{t(a)} \xrightarrow{a\cdot} P_v.$$

From this we can easily calculate

$$\mathrm{Ext}^i_{\mathbb{C}Q}(S_v, S_w) = \begin{cases} \delta_{vw}\mathbb{C}, & i = 0,\\ \mathbb{C}^{\oplus\#\{a\colon v\leftarrow w\}}, & i = 1,\\ 0, & i > 1.\end{cases}$$

If we put $S = \bigoplus_{v \in Q_0} S_v$ then $\mathrm{Ext}^{\bullet}_{\mathbb{C}Q}(S, S)$ is a graded vector space spanned by a degree 0 element $v^* = \mathbb{1}_{S_v}$ for each vertex $v \in Q_0$ and a degree 1 element a^* for each arrow $a \in Q_1$. The product structure between the degree 0 and degree 1 elements is reversed, so we can consider v^* and a^* as vertices and arrows of the opposite quiver. The product of two degree 1 elements is zero because there are no degree 2 elements. This gives the following Ext-ring:

$$\mathrm{Ext}^{\bullet}_{\mathbb{C}Q}(S, S) = \frac{\mathbb{C}Q^{\mathrm{op}}}{\langle a^* b^* \mid a^* b^* \in Q_1^{\mathrm{op}} \rangle}.$$

In analogy to the dual numbers (Example 2.34), we will call this algebra the *dual quiver algebra* and denote it by ΛQ. Because there are no degree 2 elements the higher products on ΛQ are all zero. This implies that the full subcategory $\mathtt{S}^{\bullet} \subset \mathtt{D}\,\mathtt{MOD}^{\bullet}\,Q$ of all basic simples is strictly isomorphic to ΛQ viewed as an A_∞-category with trivial higher products.

Theorem 4.5 *Let Q be any quiver; then we have the following inclusions of A_∞-categories:*

$$\mathtt{D}\,\mathtt{S}^{\bullet} \subset \mathtt{D}\,\mathtt{mod}^{\bullet}\,Q \subset \mathtt{D}\,\mathtt{P}^{\bullet}.$$

When Q has no oriented cycles these inclusions are all equivalences.

Sketch of the proof The first inclusion is obvious because the simple modules are one-dimensional. The second inclusion follows from the fact that every module M has a standard projective resolution of the form

$$\bigoplus_{a \in Q_0} Mh(a) \otimes_{\mathbb{C}} P_{t(a)} \xrightarrow{\quad d \quad} \bigoplus_{v \in Q_0} Mv \otimes_{\mathbb{C}} P_v$$

with $d(\sum_a m_a \otimes x_a) = \sum_a m_a a \otimes x_a - m_a \otimes a x_a$. If M is finite-dimensional then both terms are finite direct sums of projectives, so the resolution is a twisted complex over $\mathtt{P}^{\bullet}$.

When Q has no oriented cycles then the basic projectives are finite-dimensional because there are only a finite number of paths. Furthermore, the only simple modules are the S_v, so $\mathbb{C}Q$ and hence all the basic projectives are generated by the basic simples. This implies that $\mathtt{D}\,\mathtt{P}^{\bullet} \subset \mathtt{D}\,\mathtt{S}^{\bullet}$. □

Remark 4.6 If Q has oriented cycles then all three categories are different: $\mathtt{D}\,\mathtt{S}^{\bullet}$ will only contain the nilpotent modules (i.e. those that factor through $\mathbb{C}Q/J^k$ for some $k \geq 1$), while $\mathtt{D}\,\mathtt{mod}^{\bullet}\,Q$ does not contain those P_v for which v sits in an oriented cycle.

4.1.3 Twisted Complexes

To move easily between the three categories $\mathtt{Dmod}^\bullet\, Q$, $\mathtt{D\,P}^\bullet$, $\mathtt{D\,S}^\bullet$, we will use the notion of quiver representations.

Definition 4.7 A *dimension vector* of a quiver Q is a map $\alpha\colon Q_0 \to \mathbb{N}$ and we define the size of α as $|\alpha| = \sum_v \alpha_v$. An *$\alpha$-dimensional representation* of Q is an element

$$\rho \in \mathrm{Rep}(Q, \alpha) := \bigoplus_{a \in Q_1} \mathrm{Mat}_{\alpha_{h(a)} \times \alpha_{t(a)}}(\mathbb{C}).$$

A representation ρ can be interpreted in three different ways:

- as a right $\mathbb{C}\mathtt{Q}$-module $\mathcal{M}_\rho\colon \mathbb{C}\mathtt{Q} \to \mathtt{vect}(\mathbb{C})$ that maps the object v to the vector space $\mathcal{M}_\rho(v) := \mathbb{C}^{\alpha_v}$ and the morphism $a \in Q_1$ to the linear map $\mathcal{M}_\rho(a)\colon \mathbb{C}^{\alpha_{t(a)}} \to \mathbb{C}^{\alpha_{h(a)}} : x \mapsto x\rho(a)$, where x is considered as a row vector;
- as a twisted complex over $\mathtt{P}^\bullet$ which comes from the standard projective resolution of $\mathcal{M}_\rho$:
$$P_\rho = \bigoplus_{a \in Q_1} P_t(a)^{\oplus \alpha_{h(a)}} \xrightarrow{\rho(a)-a} \bigoplus_{v \in Q_0} P_v^{\oplus \alpha_v};$$
- as a twisted complex over $\mathtt{S}^\bullet$:
$$S_\rho := \left(\bigoplus S_v^{\oplus \alpha_v}, \delta = \sum_{a \in Q_0} \rho(a)^\top a^* \right),$$
where $\rho(a)^\top a^*$ should be seen as a matrix block between $S_{h(a)}^{\oplus \alpha_{h(a)}}$ and $S_{t(a)}^{\oplus \alpha_{t(a)}}$. The Maurer–Cartan equation is trivially satisfied because all products between the a^*'s are zero, but S_ρ is not necessarily a twisted complex because δ might not be lower triangular. This is only the case when $\mathcal{M}_\rho$ is nilpotent.

Lemma 4.8 *If $\mathcal{M}_\rho$ is nilpotent then $S_\rho \cong \mathcal{M}_\rho \cong P_\rho \in \mathtt{D\,P}^\bullet$.*

Sketch of the proof We can transform S_ρ into P_ρ by using the standard resolution of the vertex simples; P_ρ can be transformed into $\mathcal{M}_\rho$ by taking the cokernel. □

Theorem 4.9 *Every object in $\mathtt{D\,S}^\bullet$ is isomorphic to a direct sum of shifts of S_ρ's.*

Sketch of the proof We first show that every (M, δ) is isomorphic to a complex without v^*'s in δ by induction on the number of summands in M. If there is only one summand then $\delta = 0$, so the statement is trivially true.

Take a twisted complex (M, δ) and look at the bottom row of δ. By the induction hypothesis we may assume that the other rows of δ only contain dual arrows. After a conjugation (see Remark 3.31) we can ensure that at most one entry of the bottom row is a vertex $v^* = \mathbb{1}_{S_v}$.

This vertex connects two summands $S_v[i+1] \xrightarrow{\mathbb{1}} S_v[i]$. If (M, δ) is not a direct sum of two smaller complexes, all other summands are connected to $S_v[i+1]$ or $S_v[i]$ via an entry containing dual arrows, but not to both for reasons of degree. The entries leaving $S_v[i]$ or arriving in $S_v[i+1]$ are zero because $\delta^2 = 0$. Hence, up to a shift the complex looks like

$$\begin{tikzcd} N_1[1] \arrow[loop left, "\delta_1"] & S_v[1] \arrow[l, "r"'] \arrow[r, "\mathbb{1}"] & S_v & N_2 \arrow[l, "s"'] \arrow[loop right, "\delta_2"] \end{tikzcd} .$$

This complex is quasi-isomorphic to $(M_{\text{red}}, \delta_{\text{red}}) = (N_1 \oplus N_2[1], \delta_1 + \delta_2)$ via the following pair of quasi-isomorphisms:

$$\begin{tikzcd}
(M_{\text{red}}, \delta_{\text{red}}) \arrow[d, "\phi"] & N_1[1] \arrow[loop left, "\delta_1"] \arrow[d, "\mathbb{1}"] & & & N_2 \arrow[loop right, "\delta_2"] \arrow[d, "\mathbb{1}"] \arrow[dll, "-s"'] \\
(M, \delta) \arrow[d, "\phi^\dagger"] & N_1[1] \arrow[loop left, "\delta_1"] \arrow[d, "\mathbb{1}"] & S_v[1] \arrow[l, "r"'] \arrow[r, "\mathbb{1}"] & S_v \arrow[dll, "-r"'] & N_2 \arrow[l, "s"'] \arrow[loop right, "\delta_2"] \arrow[d, "\mathbb{1}"] \\
(M_{\text{red}}, \delta_{\text{red}}) & N_1[1] \arrow[loop left, "\delta_1"] & & & N_2 \arrow[loop right, "\delta_2"]
\end{tikzcd} .$$

If a twisted complex (M, δ) has only dual arrows in δ then we can write it as a direct sum of complexes for which all summands are shifted by the same degree. Such a twisted complex is of the form $S_\rho[i]$ with $\rho(a)$ the matrix of coefficients of a^*. □

Remark 4.10 More generally one can prove that every object in $\mathtt{D\,MOD}^\bullet\, Q$ is isomorphic to a direct sum of shifts of modules (a complex with zero differential). This is a well-known theorem in representation theory that follows from the fact that $\mathbb{C}Q$ is a *hereditary algebra*: the submodules of projective $\mathbb{C}Q$-modules are also projective.

4.2 Strings and Bands

Using the formalism developed above we will now look at three special quivers and interpret their representation theory geometrically as the intersection theory of lines on a surface. For most quivers it is impossible to classify all indecomposable representations. Only if the underlying graph is a Dynkin diagram or an extended Dynkin diagram can one get a complete classification. The three quivers we will examine are of this type.

4.2.1 The Linear Quiver

Let $Q = L_n$ be the linear quiver with n vertices:

$$v_n \xleftarrow{a_{n-1}} \circ \leftarrow \cdots \longrightarrow v_2 \xleftarrow{a_1} v_1 .$$

We denote the vertices from left to right by $v_1, \ldots, v_n$ and the arrows by $a_1, \ldots, a_{n-1}$. The vertex simples will be denoted by S_i and the arrows in the dual quiver by $\alpha_i = a_i^*$:

$$S_n \xrightarrow{\alpha_{n-1}} S_{n-1} \dashrightarrow S_2 \xrightarrow{\alpha_1} S_1 .$$

For each pair $1 \leq i \leq j \leq n$, there is a twisted complex

$$S_{ij} = (S_i \oplus \cdots \oplus S_j, \delta = \alpha_i + \cdots + \alpha_{j-1}),$$

which corresponds to a module that maps the vertices $v_i, \ldots, v_j$ to a one-dimensional vector space and the arrows between them to a nonzero map, or a representation ρ_{ij} with dimension vector $\epsilon_{ij} = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$ and $\rho_{ij}(a_k) = 1$ if $i \leq k < j$.

Theorem 4.11 *All indecomposable objects in* $\mathtt{D\,S}^\bullet$ *are isomorphic to shifts of the* S_{ij}.

Sketch of the proof Theorem 4.9 tells us that every indecomposable object in $\mathtt{D\,S}^\bullet$ is a shift of an S_ρ. Because there is only one α_i leaving each dual vertex and one arriving, an indecomposable S_ρ must look like

$$\left(S_i^{\oplus m_i} \oplus S_{i+1}^{\oplus m_{i+1}} \oplus \cdots \oplus S_j^{\oplus m_j}, \delta = B_i \alpha_i + \cdots + B_{j-1}\alpha_{j-1}\right),$$

where B_j is an $(m_j \times m_{j+1})$-matrix with coefficients in $\mathbb{C}$.

Decompose $\mathbb{C}^{m_k}$ as $\Im B_k \oplus R$. If $R \neq 0$ then

$$\cdots \oplus B_{k-2}B_{k-1}R \otimes S_{k-2} \oplus B_{k-1}R \otimes S_{k-1} \oplus R \otimes S_k$$

is a direct summand, so B_k must be surjective. Similarly, B_k must be injective because otherwise

$$\operatorname{Ker} B_k \otimes S_{k-1} \oplus B_{k+1}^{-1} \operatorname{Ker} B_k \otimes S_k \oplus B_{k+2}^{-1}B_{k+1}^{-1} \operatorname{Ker} B_k \otimes S_{k+1} \oplus \cdots$$

is a summand. Therefore, all B_k are invertible and we can conjugate (see Remark 3.31) them into identity matrices. If each matrix is of size $r \times r$, the twisted complex decomposes into r subcomplexes of the form S_{ij}. □

Now that we have established the indecomposable objects, the next step is to study the morphisms between them. From the definition we can deduce that

$$\mathtt{Tw}\,\mathtt{S}^\bullet(S_{ij}, S_{kl}) = \bigoplus_{\substack{i \le u \le j \\ k \le u \le l}} \mathbb{C}\mathbb{1}_{S_u} \oplus \bigoplus_{\substack{i \le v+1 \le j \\ k \le v \le l}} \mathbb{C}\alpha_v$$

with $d\alpha_v = 0$ and $d\mathbb{1}_{S_u} = \alpha_{u-1} - \alpha_u$, unless α_{u-1} or α_u are not present in $\mathtt{Tw}\,\mathtt{S}^\bullet(S_{ij}, S_{kl})$. If that is the case, these terms are deleted from the sum. If we calculate the homology we get the following table.

Lemma 4.12 *Let S_{ij}, S_{kl} be twisted complexes as above.*

Case	$\mathtt{DS}^0(S_{ij}, S_{kl})$	$\mathtt{DS}^1(S_{ij}, S_{kl})$
$i-1 \le k-1 < j \le l$	$\mathbb{C}\iota$	0
$k-1 < i-1 \le l < j$	0	$\mathbb{C}\gamma$
Otherwise	0	0

The degree 0 element is $\iota = \sum_u \mathbb{1}_{S_u}$ where the sum runs over all S_u in the overlap. The degree 1 element is $\gamma = \alpha_v$ (any of them works as they are all equal up to homology).

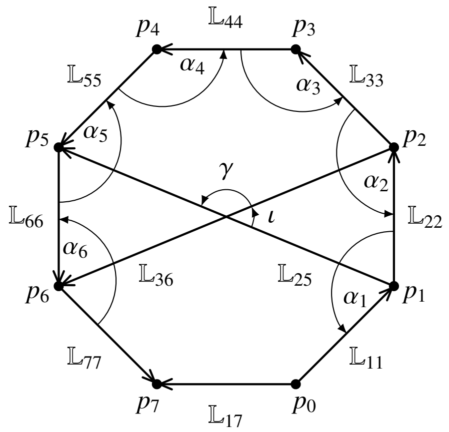
This rule can be interpreted in a geometrical way by viewing the twisted complexes as lines in a polygon.

Definition 4.13 Consider the regular $(n+1)$-gon with corners $p_0, \dots, p_n$ (we number the corners anticlockwise and assume the line $p_n p_0$ is the horizontal one at the bottom of the polygon).

A *graded line segment* is a pair $\mathbb{L} = (\overrightarrow{p_i p_j}, \theta)$ consisting of an oriented line segment that connects two corners of the polygon and a *phase* $\theta \in \mathbb{R}$. This is a number such that θ mod 2π is equal to the angle between the X-axis and $\overrightarrow{p_i p_j}$.

Every $\overrightarrow{p_i p_j}$ can be given different phases that all differ by a multiple of 2π. If we reverse the direction of a line segment we have to add π to the phase, so therefore we introduce the notation $\mathbb{L}[k] := (\overrightarrow{p_j p_i}, \theta + k\pi)$. Every graded line segment is a shift of one of the

$$\mathbb{L}_{ij} := (\overrightarrow{p_{i-1} p_j}, \theta) \quad \text{with } \theta \in (0, 2\pi].$$



If we interpret the graded line segment $\mathbb{L}_{ij}[k]$ as the representation $S_{ij}[k]$, we see that the basic simples $S_i = S_{ii}$ form the edges of the polygon except for the horizontal edge, which corresponds to S_{17} (note that the latter is oriented differently). There is a degree 1 morphism (extension) between two simples if the line segments touch. From this point of view we can see the corresponding morphisms α_i as angles of the polygon.

This also holds for other representations. Between S_{25} and S_{36} there are two morphisms: a degree 0 morphism $\iota\colon S_{25} \to S_{36}$ and a degree 1 morphism $\gamma\colon S_{36} \to S_{25}$. If we look at the picture we see that the line segments $\mathbb{L}_{25}$ and $\mathbb{L}_{36}$ intersect in the interior of the polygon and there are two angles: one in every direction.

Observation 4.14 (Morphisms are angles) *Let L_1, L_2 be two twisted complexes corresponding to graded line segments $(\mathbb{L}_1, \theta_1)$ and $(\mathbb{L}_2, \theta_2)$. If the underlying line segments are different then we have $\mathtt{D\,S}^\bullet(L_1, L_2) \neq 0$ if and only if there is a positive angle $\beta \in [0, \pi)$ inside the polygon from $\mathbb{L}_1$ to $\mathbb{L}_2$. In that case, the morphism space is generated by one morphism of degree*

$$\frac{\beta - (\theta_2 - \theta_1)}{\pi}.$$

The degree measures the difference between the turning angle and the phase shift.

This interpretation is also useful for constructing cones. If we look again at the polygon above we notice the following:

- The cone of $\alpha_i\colon S_{i+1} \to S_i$ is S_{ii+1}, which can be interpreted as the line segment formed by stitching the line segments of S_i and S_{i+1} together using the angle α_i and straightening it.

- The cone of $\gamma\colon S_{36} \to S_{25}$ is

$$\left(\underbrace{S_2 \oplus S_3 \oplus S_4 \oplus S_5}_{S_{25}} \oplus \underbrace{S_3 \oplus S_4 \oplus S_5 \oplus S_6}_{S_{36}}, \right.$$
$$\left. \underbrace{\alpha_3 + \alpha_4 + \alpha_5}_{S_{25}} + \underbrace{\alpha_4 + \alpha_5 + \alpha_6}_{S_{36}} + \underbrace{\alpha_3}_{\gamma} \right)$$

  which we can reorder as

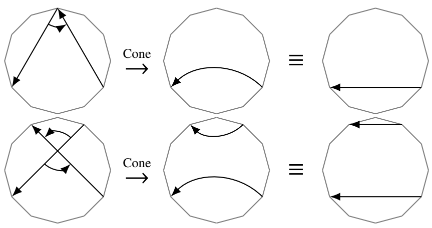
$$\left(\underbrace{S_2 \oplus \cdots \oplus S_6}_{S_{26}} \oplus \underbrace{S_3 \oplus S_4 \oplus S_5}_{S_{35}}, \underbrace{\alpha_3 + \cdots + \alpha_6}_{S_{26}} + \underbrace{\alpha_4 + \alpha_5}_{S_{35}} + \underbrace{\alpha_3}_{0 \in \mathrm{D\,S}^\bullet(S_{35}, S_{26})}\right)$$
$$\cong S_{26} \oplus S_{35}.$$

  Geometrically we used the angle γ to stitch the pieces of the line segments together differently.
- If we take the cone over $\iota\colon S_{25} \to S_{36}$ viewed as a degree 1 morphism $\iota\colon S_{25}[1] \to S_{36}$ we get

$$\left(\underbrace{S_2[1] \oplus \cdots \oplus S_5[1]}_{S_{25}[1]} \oplus \underbrace{S_3 \oplus \cdots \oplus S_6}_{S_{36}}, \right.$$
$$\left. \underbrace{-\alpha_3 - \alpha_4 - \alpha_5}_{S_{25}[1]} + \underbrace{\alpha_4 + \alpha_5 + \alpha_6}_{S_{36}} + \underbrace{\mathbb{1}_{S_3} + \mathbb{1}_{S_4} + \mathbb{1}_{S_5}}_{\iota} \right)$$

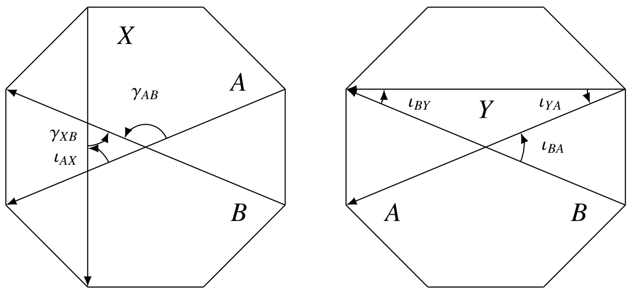
  which, using the method in the proof of Theorem 4.9, can be seen as $S_2[1] \oplus S_6$. Again we stitched the segments together using the angle.

Observation 4.15 (Cones are stitches) *Assume that $\mathbb{L}_1$, $\mathbb{L}_2$ are two graded line segments with phases in $(0, 2\pi)$ and α is an angle that corresponds to a morphism of degree 1. Taking the cone of α can be seen as the line segment obtained by stitching the two line segments over α.*

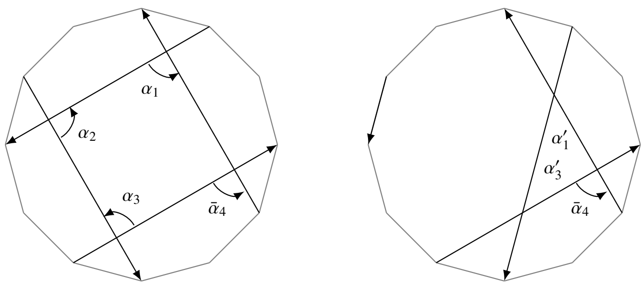


The final part of this geometrical interpretation is the products. It is easy to check that boundary angles that sit on the same corner of the polygon add up if we multiply them, but what about the internal angles? Let us look at two examples again.

- Take the twisted complexes $A = S_{36}$, $B = S_{25}$, $X = S_{47}$ and $Y = S_{35}$. There is a degree 0 morphism $\iota_{AX} \colon S_{36} \to S_{47}$ and a degree 1 morphism $\gamma_{XB} \colon S_{47} \to S_{25}$. Their composition is the degree 1 morphism $\gamma_{AB} \colon S_{36} \to S_{25}$. If we look in the picture we see that ι_{AX}, γ_{XB} are the internal angles of a triangle and γ_{AB} the external angle at the third corner.
- The product of $\iota_{BY} \colon S_{25} \to S_{35}$ and $\iota_{YA} \colon S_{35} \to S_{36}$ is $\iota_{BA} \colon S_{25} \to S_{36}$, which is the outer angle of the third corner of the triangle with inner angles ι_{BY} and ι_{YA}.



So the product sees triangles. This observation generalizes even to higher products. If we have an internal n-gon and we want to take a higher product of $n-1$ internal angles, then we can rewrite this as an $(n-2)$-ary product by taking a cone over one of the morphisms (see Lemma 3.42). The cone stitches two sides together to form an $(n-2)$-gon. Iterating this procedure we end up with a triangle that we can interpret using the binary product rule.



$$\mu(\alpha_1, \alpha_2, \alpha_3) \qquad = \qquad \mu(\alpha'_1, \alpha'_3) = \pm\bar{\alpha}_4$$

Observation 4.16 (Polygons induce products) *If $\mathbb{L}_1, \ldots, \mathbb{L}_k$ are line segments* *that bound a subpolygon then the higher product of all the connecting internal morphisms except one is the outer angle morphism at the remaining corner:*

$$\mu(\alpha_1, \ldots, \alpha_{k-1}) = \pm\bar{\alpha}_k.$$

The sign depends on the degrees of the morphisms.

Remark 4.17 Take care: not every nonzero product in $\mathtt{DS}^{\bullet}$ comes from a polygon, because sometimes these polygons can be degenerate (e.g. if three line segments run through the same point, or if some of the line segments are the same).

Observation 4.18 *The A_∞-category $\mathtt{DS}^{\bullet}$ describes the intersection theory of line segments in a polygon.*

Remark 4.19 Making this observation rigorous is not an easy task. One way to do this is to work with precategories. Define a precategory whose

- transversal sequences are graded line segments for which all pairwise intersection points are different,
- hom-spaces are graded $\mathbb{C}$-linear vector spaces spanned by the intersection angles,
- the only nonzero products are given by the rule in Observation 4.16.

Unfortunately, this precategory has no isomorphisms. To solve this problem we can add *graded curves*. These are pairs $\mathbb{L} = (\gamma\colon [0,1] \to \mathbb{R}^2, \theta\colon [0,1] \to \mathbb{R})$ where γ is a smooth embedding of $[0,1]$ into the polygon that connects two different corners and $\theta(x)$ is the angle between $\frac{d\gamma}{dt}(x)$ and the X-axis. In this precategory the angles

are quasi-isomorphisms and therefore the corresponding minimal model $\underline{\mathtt{Polygon}^{\bullet}}_{n+1}$ is a precategory with enough isomorphisms. If we categorify this precategory we get an equivalence

$$\mathtt{DS}^{\bullet} \overset{\infty}{=} \mathtt{Polygon}^{\bullet}_{n+1}.$$

4.2.2 The Kronecker Quiver

The second quiver we will have a look at is the Kronecker quiver $Q = K$. It consists of two vertices v_0, v_1 connected by two arrows a_0, a_1 in the same direction.

$$\begin{tikzcd} v_1 & v_0 \arrow[l, shift right, "a_0"'] \arrow[l, shift left, "a_1"] \end{tikzcd}$$

The category $\mathtt{P}^\bullet$ contains the two projective modules $P_i = v_i\mathbb{C}Q$ and the category $\mathtt{S}^\bullet$ contains two simple modules $S_i = v_i\mathbb{C}Q/\langle a_0, a_1\rangle$. The Ext-ring of those two simples is ΛQ. Because there are no paths of length 2, this algebra is isomorphic to $\mathbb{C}Q^{\mathrm{op}} \cong \mathbb{C}Q$. So the categories $\mathtt{S}^\bullet$ and $\mathtt{P}^\bullet$ look the same but they have different gradings. In $\mathtt{P}^\bullet$ all morphisms have degree 0 but in $\mathtt{S}^\bullet$ the dual arrows $\alpha_i = a_i^*$ have degree 1.

Theorem 4.20 *Every indecomposable object in* $\mathtt{D\,S}^\bullet$ *is isomorphic to a shift of the following complexes:*

(i) $S(i, w)$ *is a linear subcomplex of the infinite string below, which starts with* S_i *on the right and has a total of* w *terms.*

$$\cdots \xleftarrow{\alpha_1} \underbrace{S_1 \xrightarrow{\alpha_0} S_0 \xleftarrow{\alpha_1} S_1 \xrightarrow{\alpha_0} S_0}_{S(0,4)} \xleftarrow{\alpha_1} S_1 \xrightarrow{\alpha_0} \cdots$$

(ii) $B(\lambda, n) = S_0^{\oplus n} \xleftarrow{\alpha_0 \mathbb{1}_n + \alpha_1 J(\lambda,n)} S_1^{\oplus n}$, *where* $J(\lambda, n)$ *is a Jordan block of size* n *with eigenvalue* $\lambda \in \mathbb{C}^*$.

Sketch of the proof Suppose (M, δ) is an indecomposable twisted complex. Using Theorem 4.9, we can assume that the twisted complex is of the form

$$\left(S_0[k]^{\oplus n} \oplus S_1[k]^{\oplus m}, B_0\alpha_0 + B_1\alpha_1\right),$$

where B_0, B_1 are two $(m \times n)$-matrices. Up to isomorphism we can multiply both matrices by the same invertible matrix on the left and on the right. If both B_0, B_1 are invertible we can bring B_0 to the identity and then conjugate B_1 to a Jordan block. This gives us a complex of the form $B(\lambda, n)$.

If either B_0 or B_1 is not invertible we will use induction on $n + m$ to show that (M, δ) is isomorphic to an $S(i, w)$. Clearly, if $n+m = 1$ then (M, δ) is either $S_0 = S(0, 1)$ or $S_1 = S(1, 1)$. If $n + m \geq 2$ we distinguish four cases depending on whether B_i is not injective or not surjective. We work out only one of these four cases.

Suppose B_0 has a kernel. Choose a basis vector v in that kernel and let $w = B_1 v$. The latter vector is nonzero because otherwise the twisted complex is decomposable. We can extend v and w to bases for $\mathbb{C}^n$ and $\mathbb{C}^m$ such that the matrices become

$$\left(\begin{array}{c|c} 0 & \\ \vdots & B_0' \\ 0 & \end{array}\right), \quad \left(\begin{array}{c|c} 1 & \\ \vdots & B_1' \\ 0 & \end{array}\right).$$

By multiplying both on the right by an invertible matrix we can assume that B_1' starts with a zero row. The pair (B_0', B_1') satisfies the induction hypothesis because B_1' is not invertible. Hence it describes an $S(i, w)$ and the total complex looks like

$$\begin{tikzcd} \cdots & S_1 \arrow[l, "\alpha_1"'] \arrow[r, "\alpha_0"] & S_0 & \boxed{S_1} \arrow[l, "\alpha_1"'] \arrow[r, "\alpha_0"] & S_0 & S_1 \arrow[l, "\alpha_1"'] \arrow[r, "\alpha_0"] & \cdots. \\ & & & S_1 \arrow[ul, "\alpha_1"'] & & & \end{tikzcd}$$

If the bottom S_1 connects somewhere in the middle of the chain then consider the kernels K_0, K_1 of B_0, B_1 restricted to the dotted $S_1 \oplus S_1$. If they coincide then there is a direct summand isomorphic to S_1. If they span $K_0 \oplus K_1 = S_1 \oplus S_1$ then we can split the total complex in two separate complexes:

$$\begin{tikzcd} \cdots & S_1 \arrow[l, "\alpha_1"'] \arrow[r, "\alpha_0"] & S_0 & K_1 \arrow[r, "\alpha_0"] & S_0 & S_1 \arrow[l, "\alpha_1"'] \arrow[r, "\alpha_0"] & \cdots. \\ & & & K_0 \arrow[ul, "\alpha_1"'] & & & \end{tikzcd}$$

So if (M, δ) is indecomposable then that S_1 must connect to the end of the chain and hence it is a chain itself. □

Definition 4.21 We will call the $S(i, w)$ *string objects* and the $B(\lambda, n)$ *band objects*.

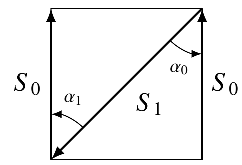
Remark 4.22 The string objects with even size can be seen as limits of band objects:

$$\lim_{\lambda \to 0} B(\lambda, n) \cong S(0, 2n) \quad \text{and} \quad \lim_{\lambda \to \infty} B(\lambda, n) \cong S(1, 2n).$$

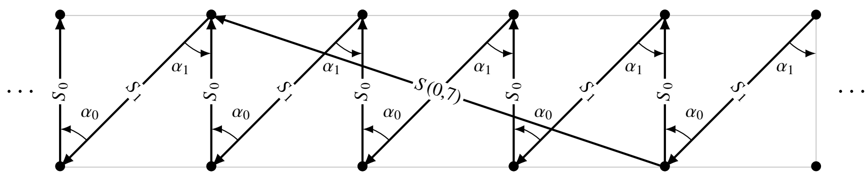
The first limit can be seen directly from the diagrams of the complexes. For the second limit, observe that a base can turn the map $\alpha_0 \mathbb{1}_n + \alpha_1 J(\lambda, n)$ into $\alpha_0 J(\lambda^{-1}, n) + \alpha_1 \mathbb{1}_n$. Therefore we will sometimes indicate these string objects as $B(0, n)$ and $B(\infty, n)$.

Now we want to obtain a similar geometrical interpretation for the Kronecker quiver as we did for the linear quiver. Take the unit square and glue the left end to the right end. We obtain a cylinder with two marked points, one on each boundary circle, coming from the corners of the square. This cylinder is a flat surface and it is possible to measure angles in the standard way by pulling back to the square. As for the polygon we can define graded line segments between the marked points.

Let S_0 denote the graded line segment that corresponds to the vertical edge of the square with phase $\frac{\pi}{2}$ and let S_1 be the downward-pointing diagonal with phase $\frac{5\pi}{4}$. There are two angles going from S_1 to S_0, which we can identify with the two degree 1 morphisms between S_1 and S_0.



In the case of the polygon, we saw that we could interpret the cone construction as gluing line segments together along an angle. In the same way, every string object $S(i, w)$ can be seen as a line segment $S(i, w)$ in the universal cover of the cylinder.



But what about the band objects? There is in fact a second type of curve with constant phase: a circle that goes around the cylinder. We would like to use these circles to interpret the band objects, but in order to do this we also need to attach some extra geometrical information: a local system.

Definition 4.23 A *local system* on a manifold $\mathbb{M}$ is a representation of its fundamental groupoid $\mathcal{L}\colon \Pi_1(\mathbb{M}) \to \mathtt{vect}(\mathbb{C})$.

A local system $\mathcal{L}$ assigns to each point of $\mathbb{L}$ a vector space $\mathcal{L}(p)$, called the fiber over p, and for each homotopy class of paths between two points there is an invertible map between the vector spaces. This map is sometimes called the transport along that path. If $\mathbb{M}$ is connected then all the vector spaces have the same dimension, which is called the *rank* of the local system.

Lemma 4.24 *The indecomposable local systems on a circle $\mathbb{S}^1$ are classified by invertible Jordan blocks.*

Sketch of the proof This is because the fundamental groupoid of the circle is equivalent to the group $\mathbb{Z}$. Up to isomorphism, a representation $\phi\colon \mathbb{Z} \to \mathsf{GL}_n$ is determined by the conjugacy class of the invertible matrix $\phi(1)$, which can be brought in Jordan normal form. □

If we fix one horizontal circle in the cylinder then each band object $B(n, \lambda)$ can be identified with a local system on that circle, for which the transport along the circle is a Jordan block of size n and eigenvalue λ. We denote the embedded circle together with the local system by $\mathbb{B}(n, \lambda)$.

The string objects can also be considered as line segments equipped with a rank 1 local system. Local systems on a contractible space are classified by their rank and a rank n local system is the direct sum of n copies of a rank 1 local system. Therefore we can interpret $S_0 \oplus S_0$ either as two line segments each with a rank 1 local system, or as one line segment with a rank 2 local system. So from now on we assume that a graded line segment $\mathbb{L}$ automatically comes with a local system and we let $\mathbb{L}(p)$ denote the fiber at p.

To work with these local systems, we also need to upgrade the morphism spaces. Every angle around an intersection point p should be seen as a morphism between the fibers of the two local systems at p. So if $\mathbb{L}_1$, $\mathbb{L}_2$ are local systems over two different line segments then we set

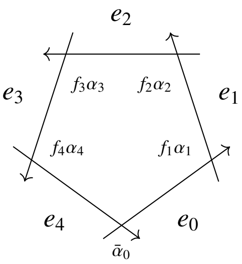
$$\mathrm{Hom}(\mathbb{L}_1, \mathbb{L}_2) = \bigoplus_{\alpha_p} \mathtt{vect}(\mathbb{L}_1(p), \mathbb{L}_2(p))\alpha_p,$$

where α_p represents an angle at the intersection point p. In other words, the angles come with coefficients that are linear maps.

This idea fits well with the cone construction. If we want to stitch two local systems together with a morphism $f\alpha_p$, where $f \in \mathtt{vect}(\mathbb{L}_1(p), \mathbb{L}_2(p))$ is invertible, we use f to identify the two fibers. If f is not invertible we break the local system in two (one where f is invertible and one where it is zero) and only stitch the bits where f becomes invertible.

The object $B(n, \lambda)$ can be seen as two local systems of rank n, one on S_0 and one on S_1, stitched together on one end by the identity and on the other end by the Jordan block. Together this gives a curve isotopic to the circle with a local system for which the transport is equal to the Jordan block: $\mathbb{B}(n, \lambda)$.

The final adaptation needed is the rule for the product. The coefficients for the angles must be included bilinearly in the products, but consecutive angles in a product do not connect fibers at the same intersection points, so the linear maps do not match up. Luckily this can easily be solved using the local system. If $e\colon p \to q$ is each edge of the polygon that lies on $\mathbb{L}_i$ then $\mathbb{L}_i(e)\colon \mathbb{L}(p) \to \mathbb{L}(q)$ is a homomorphism between the fibers on p and q. Therefore we can string all these maps together and define an extended product:



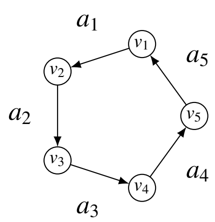
$$\mu(f_1\alpha_1, \ldots, f_4\alpha_4) = \mathbb{L}_1(e_0)f_1\mathbb{L}_1(e_1)\ldots\mathbb{L}_1(e_3)f_4\mathbb{L}_k(e_4)\cdot\bar{\alpha}_0 + \cdots.$$

If there is more than one polygon bounded by these angles, we take the sum of these expressions. With these extra features in operation one can check that the same observations we made for $\mathtt{D\,S}^\bullet$ in the case of the linear quiver also hold for the Kronecker quiver.

Observation 4.25 *The objects in* $\mathtt{D\,S}^\bullet$ *can be identified with direct sums of graded line segments with local systems. The morphisms are angles weighted by maps between the fibers of the local systems and polygons contribute to products weighted by the total transport around them.*

4.2.3 The Cyclic Quiver

The final quiver we will have a look at is the cyclic quiver $Q = C_n$. It has n vertices $v_1, \ldots, v_n$ and n arrows $a_i : v_i \to v_{i+1}$ where the index is considered modulo n.



We allow the case $n = 1$ where the quiver has one vertex and one loop. There are n projective modules $P_i = v_i\mathbb{C}Q$ in $\mathtt{P}^\bullet$ and n simple modules $S_i = v_i\mathbb{C}Q/J$ in $\mathtt{S}^\bullet$. The dual arrows $\alpha_i = a_i^*$ are degree 1 morphisms from S_{i+1} to S_i.

The situation is slightly different from the two previous cases. The algebra $\mathbb{C}Q$ is infinite-dimensional because there are paths with arbitrary length. If we take the sum of all paths of length n we get a central element in $\mathbb{C}Q$ that we will denote by

$$\ell = a_n \dots a_1 + a_1 a_n \dots a_2 + \dots + a_{n-1} \dots a_1 a_n.$$

One can easily check that $\mathbb{C}[\ell]$ is a polynomial ring and $\mathbb{C}Q$ is a free $\mathbb{C}[\ell]$ module generated by all paths of length $< n$. Because $\mathbb{C}Q$ is infinite-dimensional, the three categories $\mathtt{D\,S}^\bullet \subset \mathtt{D\,mod}^\bullet\, Q \subset \mathtt{D\,P}^\bullet$ are all different and this is reflected in the classification theorem.

Theorem 4.26 *The indecomposable objects in $\mathtt{D\,P}^\bullet$ are shifts of object of the form*

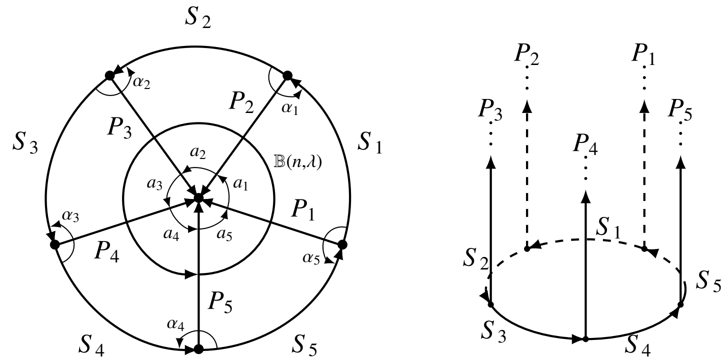
- $S(i, w) = (S_i \oplus \dots \oplus S_{i+w-1}, \alpha_i + \dots + \alpha_{i+w-1})$,
- $B(k, \lambda) = \mathcal{M}_\rho$ *with* $\rho(a_n) = J(k, \lambda)$ *and* $\rho(a_i) = \mathbb{1}_k$ *for* $i \neq n$,
- P_i,

with $i \in \{1, \dots, n\}$, $\lambda \in \mathbb{C}$ and $w \in \mathbb{N}$. We have that $S(i, w) \in \mathtt{D\,S}^\bullet$, $B(n, \lambda) \in \mathtt{D\,mod}^\bullet\, Q$ and $P_i \in \mathtt{D\,P}^\bullet$.

Sketch of the proof Every indecomposable object in $\mathtt{D\,P}^\bullet$ is a shift of a finitely generated $\mathbb{C}Q$-module M. If M is infinite-dimensional then there is an injective morphism $P_v \to M$ which splits because P_v is projective. Therefore, $M \cong P_i$.

If M is finite-dimensional, it is of the form M_ρ for some representation ρ. Because ℓ is central we can split M as a direct sum according to the eigenvalues of ℓ. Therefore, ℓ can only have one eigenvalue λ. If λ is nonzero then all $\rho(a_i)$ are invertible and we can conjugate them to the identity matrix, except for $\rho(a_n)$ which can be conjugated to a Jordan block. If $\lambda = 0$ then the module is nilpotent and hence it sits in $\mathtt{D\,S}^\bullet$. We can use the same argument as in Theorem 4.20 to complete the classification. □

Again we can interpret this classification in terms of lines on a surface. Take a disk with one marked point at the center and n on the boundary. We identify the n simple objects with the n anticlockwise arcs on the boundary and the n projective objects with the spokes. The $B(n, \lambda)$ will correspond to curves that circle around the center equipped with a local system with transport equal to $J(n, \lambda)$.



The arrows of the quiver Q correspond to morphisms between the projectives and can be seen as angles around the marked point at the center. The dual arrows run between the simples and they can be seen as angles around the marked points on the circle.

At first everything seems to work out except for the gradings; if we measure our angles in the Euclidean plane then the morphism $a_n \dots a_1$ from P_1 to itself must have degree 2 because it turns around 2π. The solution to this problem is to consider this surface as a cylinder with an infinite end instead of a disk. In that case, the line segments P_i all have phase $\frac{\pi}{2}$ and the S_i have phase 0. The triangle bounded by S_i, P_i and P_{i+1} has two angles of $\frac{\pi}{2}$ and one zero angle at infinity.

Observation 4.27 *We can interpret the representation theory of the cyclic quiver in terms of curves on a cylinder with one infinite end. The category* $\mathtt{D\,S}^\bullet$ *contains the line segments that sit on the boundary circle and* $\mathtt{D\,mod}^\bullet\,Q$ *the line segments that are compact (i.e. do not go to infinity), while* $\mathtt{D\,P}^\bullet$ *also includes the noncompact line segments.*

4.3 Points and Sheaves

In this section we will review the three quivers from a different geometrical perspective: namely algebraic geometry. We will work our way through the list in reverse order.

4.3.1 The Cyclic Quiver

Let us start with the one-loop quiver. Its path algebra is the polynomial ring $R = \mathbb{C}[X]$. This ring can be viewed as the ring of polynomial functions on the affine line $\mathbb{A}^1 = \mathbb{C}$. For each point $p \in \mathbb{C}$ there is a morphism $\rho_p \colon R \to \mathbb{C} \colon f(X) \to f(p)$ that evaluates each function at the point p. This morphism has as kernel $\mathfrak{p} = \{f \in R \mid f(p) = 0\} = (X - p)$ and if we divide out this kernel we get the simple module $S_p = J(p, 1)$. All simple modules arise in this way so we can conclude that the simple R-modules correspond to the points on the affine line:

$$\{\text{points of } \mathbb{A}^1\} \leftrightarrow \{\text{maximal ideals in } R\} \leftrightarrow \{\text{simple } R\text{-modules}\}.$$

The higher Jordan blocks can be interpreted in a similar way: while ρ_p assigns to each function its value in p, the map $R \to J(p, n) = \mathbb{C}[X]/(X - p)^n$ assigns to each f its nth-order Taylor approximation in p, so the $J(p, n)$ explore infinitesimal surroundings of p.

Given a module M we say that it is supported at $p \in \mathbb{A}^1$ if $M/\mathfrak{p}M \neq 0$. Clearly, S_p is only supported at p and at no other $q \in \mathbb{A}^1$. The same holds for $J(p, n)$, but for R as a module over itself the situation is different. It is supported at all p. At every point $R/\mathfrak{p}$ is a one-dimensional vector space and we can group all these vector spaces together into a one-dimensional vector bundle (or line bundle) over $\mathbb{A}^1$. Each element $m \in R$ gives an element in $R/\mathfrak{p}$ for every $p \in \mathbb{A}^1$ so they can be seen as sections of the bundle. The line bundle is trivial because the element $1 \in R$ gives a section that forms a basis in each fiber. Because $\mathbb{A}^1$ is contractible every vector bundle over $\mathbb{A}^1$ is trivial and if it has rank n, it will correspond to the projective (even free) module $R^{\oplus n}$:

$$\{\text{vector bundles over } \mathbb{A}^1\} \leftrightarrow \{\text{projective } R\text{-modules}\}.$$

Modules that are neither projective nor simple have a more intricate geometrical interpretation that is called a coherent sheaf. We will discuss this concept in more detail in Chapter 7, but for now it is sufficient to know that a coherent sheaf on $\mathbb{A}^1$ is the same as a finitely generated $\mathbb{C}[X]$-module:

$$\mathtt{D\,Coh}^\bullet\,\mathbb{A}^1 = \mathtt{D\,Mod}^\bullet\,\mathbb{C}[X].$$

Observation 4.28 *The representations of the one-loop quiver can be interpreted as sheaves on the affine line $\mathbb{A}^1$. Under this correspondence the simple module S_p becomes the skyscraper sheaf $\mathscr{S}_p$, while the free module $\mathbb{C}[X]$ becomes the trivial line bundle $\mathscr{O}$, also known as the structure sheaf.*

We will now extend this geometrical interpretation to the higher cyclic quivers. Consider again the affine line $\mathbb{A}^1$. On this space we have a left action of the cyclic group $G = \langle g \mid g^n \rangle \cong \mathbb{Z}_n$:

$$g^k \cdot x = \zeta^k x \quad \text{where} \quad \zeta = e^{\frac{2\pi i}{n}}.$$

The orbits for this action all have size n except for the zero orbit, which consists of one point. The action transfers to a right action on the coordinate ring $\mathbb{C}[X]$ such that $f \cdot g(x) = f(g \cdot x)$. If we want to quotient out the action, the coordinate ring of the quotient $\mathbb{A}^1/G$ should be seen as the ring of G-invariant functions $\mathbb{C}[X]^G$, because these are the functions that are constant on the orbits. This ring is $\mathbb{C}[X^n]$, which is isomorphic to $\mathbb{C}[X]$, so from this perspective $\mathbb{A}^1/G$ looks the same as $\mathbb{A}^1$.

However, this does not fully capture the geometry, because $\mathbb{A}^1/G$ has one special point (the zero), while all points of $\mathbb{A}^1$ are alike. Therefore, it is better to describe the quotient not as an affine variety but as something more delicate: the orbifold $[\mathbb{A}^1/G]$.

To do this we introduce the notion of a G-equivariant sheaf. This is a sheaf on $\mathbb{A}^1$ together with an action of G that is compatible with the action of G on $\mathbb{A}^1$. To be more precise, it is a $\mathbb{C}[X]$-module M with a G-action such that $(mf(x)) \cdot g = (m \cdot g)(f(x) \cdot g)$. Both actions can be packaged together in one algebra

$$\mathbb{C}[X] \star G := \frac{\mathbb{C}\langle X, g\rangle}{\langle gXg^{-1} - \zeta^k X, g^n - 1\rangle},$$

and then M naturally gets the structure of a $\mathbb{C}[X] \star G$-module.

Therefore, it makes sense to define the derived category of G-equivariant coherent sheaves as

$$\mathtt{D}\,\mathtt{Coh}^\bullet[\mathbb{A}^1/\mathbb{Z}_n] := \mathtt{D}\,\mathtt{Mod}^\bullet\,\mathbb{C}[X] \star G.$$

This category is the right substitute for sheaves on $[\mathbb{A}^1/G]$, so $\mathtt{D}\,\mathtt{Coh}^\bullet[\mathbb{A}^1/G]$ should be considered as the derived category of coherent sheaves on the orbifold $[\mathbb{A}^1/G]$.

Theorem 4.29 *The algebra $\mathbb{C}[X] \star G$ is isomorphic to $\mathbb{C}Q$ with Q the cyclic quiver with n vertices.*

Sketch of the proof Put $\zeta = e^{\frac{2\pi i}{n}}$ and define

$$e_k = \frac{1}{n}\sum_{i=1}^{n} \zeta^{ki} g^i.$$

It is easy to check that $\sum_k e_k = 1$ and $e_i e_j = \delta_{ij} e_i$. In other words, they form a set of orthogonal idempotents. Moreover, we have $Xe_k = e_{k+1}X$ and

$$X = \sum_i e_{i+1} X e_i,$$

where the indices are taken modulo n. Now construct a morphism $\phi\colon \mathbb{C}Q \to$

$\mathbb{C}[X] \star G$ that maps the vertices to the e_k and the arrows a_i to the $e_{i+1}Xe_i$. One can easily check that this is an isomorphism. ◻

Observation 4.30 *The representation theory of the cyclic quiver can be geometrically interpreted as coherent sheaves on an orbifold:*

$$\mathtt{D}\,\mathtt{Coh}^\bullet[\mathbb{A}^1/G] \overset{\infty}{=} \mathtt{D}\,\mathtt{Mod}^\bullet\,\mathbb{C}C_n.$$

4.3.2 The Kronecker Quiver

To get a similar interpretation for the Kronecker quiver, we have to look at the algebraic geometry of the projective line. This is the affine line with an extra point at infinity: $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$. Using homogeneous coordinates we can also see it as

$$\mathbb{P}^1 := \{(x : y) \mid (x, y) \in \mathbb{C}^2 \setminus \{(0, 0)\}\},$$

where $(x : y)$ is considered up to a multiple: $(x : y) = (\lambda x : \lambda y)$. We can see it as the union of two affine lines, $\{(x : 1) \mid x \in \mathbb{C}\}$ and $\{(1 : y) \mid y \in \mathbb{C}\}$, whose intersection is $\mathbb{C}^*$.

The coordinate rings of the two affine lines are $R_X = \mathbb{C}[X]$ and $R_Y = \mathbb{C}[Y]$. Both X and Y can be considered as functions on the overlap and there they satisfy the relation $Y = \frac{1}{X}$. The coordinate ring on the overlap can be seen as $R_{XY} = \frac{\mathbb{C}[X,Y]}{(XY-1)}$. On the other hand, the only functions in $\mathbb{C}[X]$ and $\mathbb{C}[Y]$ that extend to $\mathbb{P}^1$ are the constant ones. We can summarize this in two diagrams: one of spaces and one of algebras.

$$\begin{tikzcd}
& \mathbb{P}^1 & \\
\mathbb{A}^1 \arrow[ur, hook] & & \mathbb{A}^1 \arrow[ul, hook'] \\
& \mathbb{C}^* \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{C} \arrow[dl, hook'] \arrow[dr, hook] & \\
R_X \arrow[dr, hook] & & R_Y \arrow[dl, hook'] \\
& R_{XY} &
\end{tikzcd}$$

The ring of functions on $\mathbb{P}^1$ is just $\mathbb{C}$, which does not fully capture the geometry of the projective line. Therefore we cannot view sheaves on $\mathbb{P}^1$ as modules over its coordinate ring.

The solution to this problem is to view a coherent sheaf over $\mathbb{P}^1$ as two coherent sheaves over the two $\mathbb{A}^1$'s glued together. This means that we need three ingredients: a finitely generated R_X-module M_X, a finitely generated R_Y-module M_Y and a gluing isomorphism between them on the overlap. The latter is an isomorphism of R_{XY}-modules:

$$\phi_M\colon \underbrace{M_X \otimes_{R_X} R_{XY}}_{M_{XY}} \to \underbrace{M_Y \otimes_{R_Y} R_{XY}}_{M_{YX}}.$$

We call such a triple $M = (M_X, M_Y, \phi_M)$ a coherent sheaf on $\mathbb{P}^1$.

The morphisms in $\mathsf{Coh}\,\mathbb{P}^1$ are the expected ones: pairs of module morphisms $(\psi_X\colon M_X \to N_X, \psi_Y\colon M_Y \to N_Y)$ that commute with the gluing maps. There is a neat way to see these morphisms as the zeroth homology of the complex

$$\begin{pmatrix}\mathrm{Hom}_{R_X}(M_X, N_X)\\ \oplus\\ \mathrm{Hom}_{R_Y}(M_Y, N_Y)\end{pmatrix} \xrightarrow{d_0} \mathrm{Hom}_{R_{XY}}(M_{XY}, N_{XY})$$

with $d_0(\psi_X \oplus \psi_Y) = \psi_X \otimes_{R_X} R_{XY} - \phi_N^{-1} \circ (\psi_Y \otimes_{R_Y} R_{XY}) \circ \phi_M$.

Theorem 4.31 *Every coherent sheaf is a direct sum of the following types:*

(i) $\mathscr{S}(\lambda, n) := (\frac{\mathbb{C}[X]}{(X-\lambda)^n}, \frac{\mathbb{C}[Y]}{(Y-\lambda^{-1})^n}, \mathbb{1})$,
(ii) $\mathscr{S}(0, n) := (\mathbb{C}[X]/(X^n), 0, 0)$,
(iii) $\mathscr{S}(\infty, n) := (0, \mathbb{C}[Y]/(Y^n), 0)$,
(iv) $\mathscr{O}(k) := (\mathbb{C}[X], \mathbb{C}[Y], Y^k)$,

where $\lambda \in \mathbb{C}^*$, $n \in \mathbb{N}$ and $k \in \mathbb{Z}$.

Sketch of the proof Decompose M_X as a direct sum of indecomposable modules. Its components are either of the form $R_X/(X-\lambda)^n$ with $\lambda \in \mathbb{C}$ or the free module R_X. If we tensor with R_{XY} the latter stays free, while $R_X/(X-\lambda)^n$ becomes $R_{XY}/(X-\lambda)^n$. In the case $\lambda \neq 0$ this module remains n-dimensional, but for $\lambda = 0$ it becomes zero. We can reason the same way for M_Y but note that $R_{XY}/(Y-\lambda)^n$ can be rewritten as $R_{XY}/(X-\lambda^{-1})^n$.

Pairing up isomorphic components we see that every $\mathbb{C}[X]/(X-\lambda)^n$ in M_X must be balanced by a $\mathbb{C}[Y]/(Y-\lambda^{-1})^n$ in M_Y and every $\mathbb{C}[X]$ by a $\mathbb{C}[Y]$. Only the $\mathbb{C}[X]/(X^n)$ in M_X and the $\mathbb{C}[Y]/(Y^n)$ in the M_Y do not need to be balanced and they can be split off as irreducible sheaves of type $\mathscr{S}(0, n)$ or $\mathscr{S}(\infty, n)$.

The rest needs to be glued together by an isomorphism. For example, if $M_{XY} = R_{XY}^{\oplus n}$ then this automorphism is an invertible $(n \times n)$-matrix G with coefficients in R_{XY}. By applying automorphisms of $M_X = R_X^{\oplus n}$ and $M_Y = R_Y^{\oplus n}$ this matrix is only defined up to multiplication on the right with invertible matrices over R_X and on the left with invertible matrices over R_Y. A classical theorem by Dedekind and Weber tells us that we can bring G into diagonal form with integer powers of Y on the diagonal, so in that case the sheaf is a direct sum of $\mathscr{O}(k)$'s.

On the other hand, morphisms between R_{XY}-modules of the form $R_{XY}/(X -$

$\lambda)^n$ all come from R_X-morphisms. Therefore we can assume that the gluing between these components is just the identity, so we get direct sums of $\mathscr{S}(\lambda, n)$'s. For more details see [Gro57, Sor, Har77, HM82, str]. □

Remark 4.32 The $\mathscr{S}_\lambda := \mathbb{S}(\lambda, 1)$ are the *skyscraper sheaves* and the $\mathscr{O}(k)$ are the *line bundles* over $\mathbb{P}^1$. The sheaf $\mathscr{O} := \mathscr{O}(0)$ is the trivial line bundle or structure sheaf.

In this way we have constructed an ordinary category of coherent sheaves, but we want to upgrade it to an A_∞-category. Just as we substituted every module with a free resolution, we can do the same here but now we need locally free resolutions. These are sheaves of the form (M_X, M_Y, ϕ_M) where both M_X, M_Y are free. It is easy to check that every coherent sheaf has a locally free resolution, which we can use to make dg-hom-spaces.

There is however one extra feature: as we saw earlier the sheaf homomorphisms themselves form the zeroth homology of a complex. Therefore we have two differentials and by taking their sum we get one big complex. The morphism spaces in $\mathsf{Coh}^\bullet\,\mathbb{P}^1$ are the cohomologies of these complexes. The degree 0 parts are the ordinary morphisms of sheaves, while the higher-degree part measures both the algebraic extensions between the modules as the geometrical obstructions in the gluing process.

Example 4.33 The dg-hom-space between $\mathscr{O}(r)$ and $\mathscr{O}(s)$ is

$$\underbrace{\mathrm{Hom}_{R_X}(R_X, R_X) \oplus \mathrm{Hom}_{R_Y}(R_Y, R_Y)}_{\text{degree 0}} \oplus \underbrace{\mathrm{Hom}_{R_{XY}}(R_{XY}, R_{XY})}_{\text{degree 1}}$$

with $d(f(X) \oplus g(Y)) = f(X) - X^{s-r} g(Y)$.

If $r > s$ then the second term can never cancel the first because it only consists of negative powers of X. Therefore, the zeroth homology is zero. If $r \le s$ then we have the following basis for the zeroth homology $(f, g) = (X^i, Y^{s-r-i})$ with $0 \le i \le s - r$:

$$\mathsf{Coh}^0(\mathscr{O}(r), \mathscr{O}(s)) = \begin{cases} \mathbb{C}(1, Y^{s-r}) + \cdots + \mathbb{C}(X^{s-r}, 1), & r \le s, \\ 0, & r > s. \end{cases}$$

The first homology on the other hand consists of $\mathbb{C}[X, X^{-1}]$ divided by the vector space spanned by X^i and X^{s-r-i} with $i \ge 0$. If $s < r - 1$ this space is nonzero and has dimension $r - s - 1$:

$$\mathsf{Coh}^1(\mathscr{O}(r), \mathscr{O}(s)) = \begin{cases} 0, & r \le s + 1, \\ \mathbb{C}Y + \cdots + \mathbb{C}Y^{r-s-1}, & r > s + 1. \end{cases}$$

Theorem 4.34 *The derived A_∞-category of coherent sheaves* $\mathtt{D\,Coh}^\bullet\,\mathbb{P}^1$ *is A_∞-equivalent to* $\mathtt{D\,mod}^\bullet\,\mathbb{C}Q$ *where $Q = K_2$ is the Kronecker quiver.*

Sketch of the proof We can construct all sheaves using $\mathscr{O}(0)$ and $\mathscr{O}(1)$:

- $\mathscr{O}(i-1)$ is the kernel of $\mathscr{O}(i)^{\oplus 2} \xrightarrow{(1\ Y)} \mathscr{O}(i+1)$,
- $\mathscr{O}(i+1)$ is the cokernel of $\mathscr{O}(i-1) \xrightarrow{\binom{1}{Y}} \mathscr{O}(i)^{\oplus 2}$,
- $\mathscr{S}(\frac{\lambda}{\mu}, n)$ is the cokernel of $\mathscr{O}(0) \xrightarrow{(\lambda+\mu Y)^i} \mathscr{O}(n)$.

Therefore, $\mathtt{D\,Coh}^\bullet\,\mathbb{P}^1 = \langle \mathscr{O}(0), \mathscr{O}(1)\rangle$. Take $\mathscr{O}(0) \oplus \mathscr{O}(1)$ and look at its A_∞-endomorphism algebra. Using the calculation in Example 4.33 we can conclude that

$$\mathtt{Coh}\,(\mathscr{O}(0) \oplus \mathscr{O}(1), \mathscr{O}(0) \oplus \mathscr{O}(1)) = \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{C} + \mathbb{C}Y & \mathbb{C} \end{pmatrix}$$

concentrated in degree 0. This is isomorphic to the path algebra of the Kronecker quiver, so $\langle \mathscr{O}(0), \mathscr{O}(1)\rangle = \mathtt{D}\mathbb{C}Q = \mathtt{D\,mod}^\bullet\,\mathbb{C}Q$. □

Remark 4.35 In this identification the string objects of odd length correspond to the line bundles $\mathscr{O}(k)$, the string objects of even length with the $\mathscr{S}(0, n)$ and $\mathscr{S}(\infty, n)$ and the band objects with the $\mathscr{S}(\lambda, n)$.

Observation 4.36 *The representation theory of the Kronecker quiver can be geometrically interpreted as coherent sheaves on the projective line:*

$$\mathtt{D\,Coh}^\bullet\,\mathbb{P}^1 \overset{\infty}{\cong} \mathtt{D\,Mod}^\bullet\,K_2.$$

4.3.3 The Linear Quiver

For the interpretation of the linear quiver we need the notion of a *Landau–Ginzburg model*. This consists of a pair $(\mathbb{X}, f)$ where $\mathbb{X}$ is a space (an affine variety or an orbifold) and $f\colon X \to \mathbb{C}$ is a function. To this pair we can associate a curved algebra (R, f) where R is the coordinate ring of $\mathbb{X}$ and f is interpreted as an element in R.

The *category of singularities* of the Landau–Ginzburg model $(\mathbb{X}, f)$ is defined as the category of matrix factorizations. We will see in Chapter 7 that it gives a description of the geometry of the singular locus of $\mathbb{Y} = f^{-1}(0)$. Therefore, it is also denoted by

$$\mathtt{DSing}^\pm\mathbb{Y} = \mathtt{DMF}^\pm(R, f).$$

We will consider the case of the orbifold $\mathbb{X} = [\mathbb{A}^1/G]$. As we saw previously the correct notion for its coordinate ring is the algebra $\mathbb{C}[X] \star G$, which is isomorphic to the path algebra $\mathbb{C}Q$ of the cyclic quiver. This algebra has a central element $\ell = X^n$ which acts on each projective P_v as multiplication with the path of length n that starts and ends at v. This action gives an element $\mu_0(1) \in \mathrm{Hom}(P_v, P_v)$ which we can use to turn the $\mathbb{Z}_2$-graded version of $\mathtt{P}^\bullet$ into a curved category $\mathtt{P}^\pm_\ell$. The objects in $\mathtt{DP}^\pm_\ell$ consists of pairs (P, d) of a $\mathbb{Z}_2$-graded projective $\mathbb{C}Q$-module and a degree 1 map $d \colon P \to P$ such that d^2 acts as multiplication by ℓ.

Theorem 4.37 *Every indecomposable object in $\mathtt{DP}^\pm_\ell$ is a twisted complex of the form*

$$M_{ij} = \left(P_i \oplus P_j[1], \begin{pmatrix} 0 & p_{ij} \\ p_{ji} & 0 \end{pmatrix}\right),$$

where p_{uv} is the shortest path from v to u.

Sketch of the proof Let (P, d) be any matrix factorization. We split d into two parts $d_{10} + d_{01}$ that run between the degree 0 and degree 1 parts. Because $\mathrm{Hom}(P_v, P_w) = p_{wv}\mathbb{C}[\ell]$ and $\mathbb{C}[\ell]$ is a principal ideal domain, we can choose a basis in P^0 and P^1 such that d_{01} becomes a diagonal matrix. The fact that $d_{01}d_{10} = \ell$ implies that d_{10} is also diagonal. If (P, d) is indecomposable, d_{01} is a (1×1)-matrix and hence of the form M_{ij}. □

Now let us have a brief look at the hom-spaces:

$$\mathtt{TwP}^\pm_\ell(M_{ij}, M_{kl}) = \begin{pmatrix} p_{ki}\mathbb{C}[\ell] & p_{kj}\mathbb{C}[\ell] \\ p_{li}\mathbb{C}[\ell] & p_{lj}\mathbb{C}[\ell] \end{pmatrix}$$

with $d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & p_{kl} \\ p_{lk} & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}\begin{pmatrix} 0 & p_{ij} \\ p_{ji} & 0 \end{pmatrix}$. When $i = j$ or $k = l$ the homology is always zero, so M_{ii} is a zero object. Furthermore, we may assume that $i < j$ and $k < l$ because $M_{ij} = M_{ji}[1]$. Note that the indices are in fact cyclic but now we assume that they run from 0 to $n - 1$. With these assumptions we get the following table, with $\iota = \begin{pmatrix} p_{ki} & 0 \\ 0 & p_{lj} \end{pmatrix}$ and $\gamma = \begin{pmatrix} 0 & p_{kj} \\ p_{li} & 0 \end{pmatrix}$.

Case	$\mathtt{DP}_\ell{}^0(M_{ij}, M_{kl})$	$\mathtt{DP}_\ell{}^1(M_{ij}, M_{kl})$
$i \le k < j \le l$	$\mathbb{C}\iota$	0
$k < i \le l < j$	0	$\mathbb{C}\gamma$
Otherwise	0	0

This is exactly the same table as for the linear quiver of length $n-1$ if we make the identifications $M_{ij} \leftrightarrow S_{i+1j}$.

Observation 4.38 *The representation theory of the linear quiver can be geometrically interpreted as the category of singularities of a Landau–Ginzburg* *model*

$$\mathtt{DMF}^{\pm}([\mathbb{A}^1/\mathbb{Z}_n], X^n) \overset{\infty}{\cong} \mathtt{DMod}^{\pm} L_{n-1}.$$

4.4 Picturing the Categories

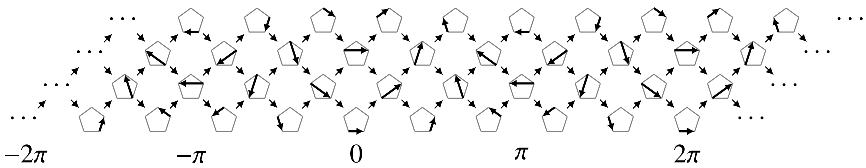
In this section we will draw the categories $\mathtt{D\,Mod}^{\bullet}\, Q$ for the three quivers we have studied. In each case we draw all the indecomposable objects and we order them on the X-axis according to their phase. We also draw the indecomposable degree 0 angle morphisms. These are those angles that cannot be written as a product of smaller angle morphisms. Because angles are always positive they will point in the positive X-direction.

The resulting picture is also known as the Auslander–Reiten quiver of the category and has been studied in detail in the representation theory of finite-dimensional algebras [ARS97]. The picture for $\mathtt{D\,Coh}^{\bullet}\, \mathbb{P}^1$ can be found in [Kel07].

For each category we will also discuss its symmetries. These symmetries are a consequence of the geometric interpretations and can be seen as A_∞-functors of the categories. The different geometrical interpretations will often give rise to different symmetries.

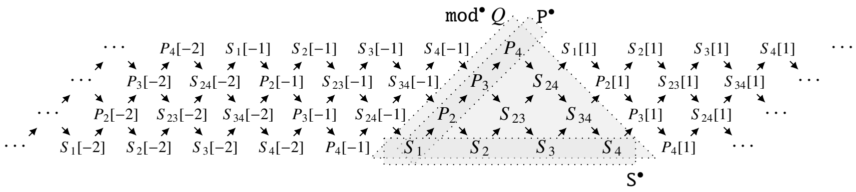
4.4.1 The Linear Quiver

Using the interpretation as graded line segments, the diagram below shows the category $\mathtt{D\,S}^{\bullet} = \mathtt{D\,P}^{\bullet}$.



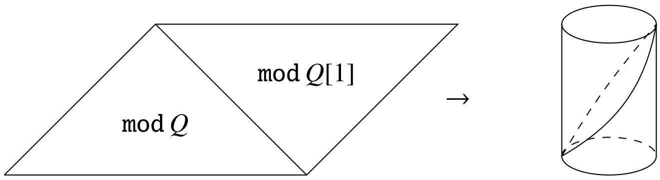
We ordered the objects in a grid using the phase for the horizontal axis and the anticlockwise difference between the source and target of the graded line segment for the vertical axis. Therefore, the shift operator is a glide reflection: it adds π to the phase and swaps source and target so it is a translation over π followed by a horizontal reflection.

We can draw the same picture for the derived category of representations of the A_n-quiver and in it we can easily identify the subcategories $\mathtt{S}^{\bullet}$, $\mathtt{mod}^{\bullet}\, Q$ and $\mathtt{P}^{\bullet}$.



The geometrical interpretation shows that this category has a special autoequivalence $\mathcal{R}$. This autoequivalence rotates all the line segments over $\frac{2\pi}{n+1}$ and adds the same angle to all the phases. This autoequivalence is obvious from the geometric point of view but highly nontrivial if we interpret the category as $\mathtt{D}\,\mathtt{mod}^\bullet\, Q$. This functor maps some representations to others like S_i, which is mapped to S_{i-1} if $i > 2$. However, S_1 is mapped to the shift of a representation: $P_n[1]$. In representation theory this functor is also known as the *Auslander–Reiten translate* (or its inverse depending on the direction of the rotation) and it corresponds to a horizontal translation over $\frac{2\pi}{n+1}$ in the diagram.

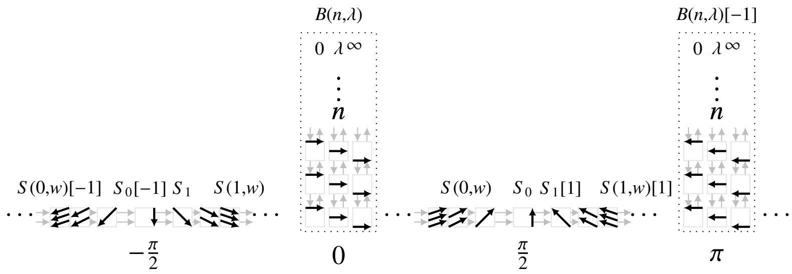
Remark 4.39 For the second geometrical interpretation we only used the $\mathbb{Z}_2$-graded version. This means that we can forget about the phases of the line segments, and the category $\mathtt{DS}^\pm$ is a rolled up version of $\mathtt{D}\,\mathtt{S}^\bullet$ that only contains two copies of $\mathtt{mod}\, Q$, which are glued together to form a cylinder.



4.4.2 The Kronecker Quiver

For the Kronecker quiver all curves with a horizontal phase are of the form $B(-, n)$, so if we set them on a diagram with as horizonal axis the phase, they all sit on the same spot. To distinguish them we will put them in a box with width parametrized by λ and height by n. In this way we get a tower of objects on the locations where the phase is a multiple of π. Each object in the tower is connected to the one above it by the injection $B(\lambda, n) \subset B(\lambda, n + 1)$ and to the one below it by the projection $B(\lambda, n) \to B(\lambda, n - 1)$. There are no degree 0 morphisms between band objects with different λ.

The string objects with odd size all have different phases: for $S(0, 2k+1)$ the phase is $\frac{\pi}{2} - \tan^{-1}(k)$, while for $S(1, 2k+1)$ it is $-\frac{\pi}{2} + \tan^{-1}(k+1)$. Each of these objects is connected to the next by two degree 0 morphisms coming from the two embeddings (one on the left and one on the right). The subcategory $\mathtt{mod}^\bullet\, Q$ consists of everything with phases in $(-\frac{\pi}{2}, \frac{\pi}{2}]$.

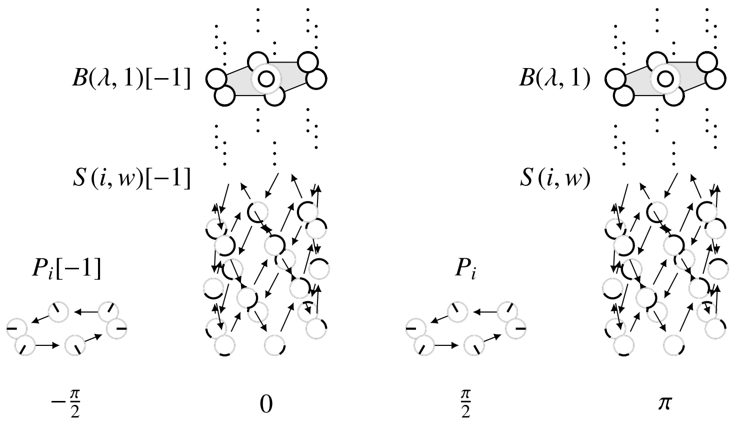


The picture indicates a symmetry of the category: we can shift all the odd string objects 1 to the right. On the cylinder this corresponds to a Dehn twist: we turn the upper boundary circle one full turn to the left, as if trying to open a jar. This gives every curve that crosses the cylinder an extra twist: $S(1, w)$ becomes $S(1, w + 2)$, while $S(0, w)$ becomes $S(0, w - 2)$ and S_0 turns into $S_1[1]$. Again we have an equivalence that is natural from the curve perspective but highly nontrivial from the quiver point of view.

A second symmetry comes from the geometrical interpretation as sheaves on $\mathbb{P}^1$. From this point of view every element $g \in \mathsf{PGL}_2(\mathbb{C})$ induces a transformation of the projective line that changes the sheaf $\mathscr{S}(p, n)$ to $\mathscr{S}(g \cdot p, n)$. The locally free sheaves are unaffected by these transformations. While obvious from the $\mathbb{P}^1$ point of view, this symmetry is very unnatural from the marked cylinder point of view because it mixes strings and band objects.

4.4.3 The Cyclic Quiver

In this picture there are only two phases up to shift. At $\frac{\pi}{2}$ we find all the projectives connected cyclically by the morphisms a_i. At 0 we find all the finite-dimensional modules, which we will order vertically according to their dimension. The string modules $S(i, w)$ can be ordered cyclically according to their starting point. In this way we get a tube with, at each height n, modules connected to the next and previous levels by a rhombic mesh. At heights that are multiples of n we find a floor of band objects.



Again there are symmetries. The first one maps $S(i, w) \to S(i+1, w)$ and $P_i \to P_{i+1}$ but fixes all the $B(\lambda, k)$. It corresponds to a rotation of the marked cylinder over $\frac{2\pi}{n}$. The second symmetry fixes the $S(i, w)$ and P_i but rescales the $B(\lambda, k)$ to $B(r\lambda, k)$. This corresponds to the rescaling of the orbifold $[\mathbb{A}^1/G]$ by a factor r.

4.5 A First Glimpse of Homological Mirror Symmetry

In this chapter we have seen that the representation theory of certain quivers can be interpreted geometrically in two different ways.

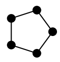
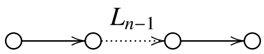
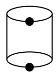
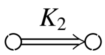
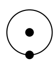
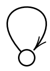
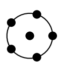
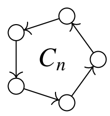
- The first interpretation has to do with the intersection theory of curves on a surface and gives rise to an A_∞-category where the objects are embedded curves and the morphisms are linear combinations of (angles at) intersection points. These categories are baby examples of a larger class of categories called Fukaya categories.

  In general, Fukaya categories describe the intersection theory of certain n-dimensional submanifolds, called Lagrangian submanifolds, of $2n$-dimensional symplectic manifolds. Just like for curves on a surface, two Lagrangian submanifolds will in general intersect at points and hence we can take the intersection points as a basis for the morphisms. There are many different versions of such categories: Fukaya categories, wrapped Fukaya categories, Fukaya–Seidel categories. Such constructions are often called *A-models* because they are related to type IIA superstring theory in physics.
- The second interpretation has to do with coherent sheaves on complex algebraic varieties. These are generalizations of vector bundles and the mor-

phisms measure maps between them (in degree 0), obstructions to the existence of maps and extensions between sheaves (in higher degrees). Again there are many different flavors of such categories: the derived category of coherent sheaves, categories of matrix factorizations, categories of singularities, etc. These constructions are called *B-models* because they appear in type IIB string theory.

The different models (quiver, A-model and B-model) give A_∞-categories that are not precisely equivalent but they become equivalent after we derive them. This procedure adds more objects to the categories and makes them nicer in a certain way without changing their representation theory.

Below is a list of models we have matched so far. This seems a small list but it is just the tip of the iceberg. It turns out that, with some creativity, it is possible to come up with a suitable B-model for almost any A-model and vice versa. This phenomenon was predicted by superstring theory and now goes under the name of *Homological Mirror Symmetry*. In the next part we are going to examine this idea in detail.

A-model	Quiver	B-model
Filled n-gon 	Linear quiver 	Curved orbifold $([\mathbb{A}^1/\mathbb{Z}_n], X^n)$
Doubly marked cylinder 	Kronecker quiver 	Projective line $\mathbb{P}^1$
Punctured 1-marked disk 	One-loop quiver 	Affine line $\mathbb{A}^1$
Punctured n-marked disk 	Cyclic quiver 	Orbifold $[\mathbb{A}^1/\mathbb{Z}_n]$