1

Smooth versus singular

1.1 A crash course in algebraic geometry

In algebraic geometry we study the connections between algebraic varieties, which are sets of solutions of polynomial equations, and complex algebras.

An *affine variety* is a subset $\mathbb{X} \subset \mathbb{C}^n$ that is defined by a finite set of polynomial equations.

$$\mathbb{X} := \{x \in \mathbb{C}^n | f_1(x) = 0, \dots, f_k(x) = 0\}$$

A *morphism between two varieties* $\mathbb{X} \in \mathbb{C}^n$ and $\mathbb{Y} \in \mathbb{C}^m$ is a map $\phi : \mathbb{X} \to \mathbb{Y}$ such that there exists a polynomial map $\Phi : \mathbb{C}^n \to \mathbb{C}^m$ with $\phi = \Phi|_{\mathbb{X}}$. Such a morphism is an isomorphism if ϕ is invertible and ϕ^{-1} is also a morphism. The affine varieties with their morphisms form a category which we will denote by $\mathtt{Aff} - \mathtt{var}$.

We can consider $\mathbb{C}$ as a variety, so it makes sense to look at the morphisms from a variety $\mathbb{X}$ to $\mathbb{C}$, these maps are also called the regular functions on $\mathbb{X}$. They are closed under pointwise addition and multiplication so they form a commutative $\mathbb{C}$-algebra: $\mathbb{C}[\mathbb{X}]$.

This algebra can be described with generators and relations. To every variety $\mathbb{X} \subset \mathbb{C}^n$ the set of polynomial functions that are zero on $\mathbb{X}$ form an ideal in $\mathbb{C}[x_1, \cdots, x_n]$. If we divide out this ideal we get the ring of polynomial functions on $\mathbb{X}$.

$$\mathbb{C}[\mathbb{X}] := \mathbb{C}[X_1, \ldots, X_n]/(f|\forall x \in \mathbb{X} : f(x) = 0)$$

This algebra is finitely generated by the x_i and it also has no nilpotent elements because $f(x)^n = 0 \Rightarrow f(x) = 0$.

Take care: if $\mathbb{X} = \{x \in \mathbb{C}^n | f_1(x) = \ldots, = f_k(x)\}$ then it is not necessarily true that $\mathbb{C}[\mathbb{X}]$ is isomorphic to $\mathbb{C}[X_1, \ldots, X_n]/(f_1, \ldots, f_k)$ because this algebra might have nilpotent elements. To solve this we need to define the *radial ideal* of $(f_1, \ldots, f_k)$:

$$\sqrt{(f_1, \ldots, f_k)} := \{h \in \mathbb{C}[X_1, \ldots, X_n] | \exists j \in \mathbb{N} : h^j \in (f_1, \ldots, f_k)\}.$$

With this notation we have that

Theorem 1.1 (Hilbert Nullstellensatz)**.** *If* $\mathbb{X} = \{x \in \mathbb{C}^n | f_1(x) = \ldots, = f_k(x)\}$ *then*

$$\mathbb{C}[\mathbb{X}] \cong \frac{\mathbb{C}[X_1, \ldots, X_n]}{\sqrt{(f_1, \ldots, f_k)}}.$$

If R is a finitely generated commutative $\mathbb{C}$-algebra without nilpotent elements, we will call this an *affine algebra*. The affine algebras together with algebra morphisms form a category, which we denote by $\texttt{Aff} - \texttt{alg}$.

A morphism between varieties, $\phi : \mathbb{X} \to \mathbb{Y}$, will also give an algebra morphism between the corresponding rings but the arrow will go in the opposite direction:

$$\phi^* : \mathbb{C}[\mathbb{Y}] \to \mathbb{C}[\mathbb{X}] : g \mapsto g \circ \phi.$$

In light of the previous section we can say that operation $\mathbb{C}[-]$ defines a contravariant functor from $\texttt{Aff} - \texttt{var}$ to $\texttt{Aff} - \texttt{alg}$.

Theorem 1.2 (Main theorem of affine geometry)**.** *The functor* $\mathbb{C}[-]$ *defines an anti-equivalence between* $\texttt{Aff} - \texttt{var}$ *and* $\texttt{Aff} - \texttt{alg}$*. So working with affine varieties is actually the same as working with affine algebras but with all maps are reversed.*

The anti-equivalence implies that we can also go in the opposite direction: from algebras to varieties. Because $R \in \texttt{Aff} - \texttt{alg}$ is finitely generated, it can be written as a quotient $\mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_k)$. Therefore we can associate to R the variety $\mathbb{V}(R)$ in $\mathbb{C}^n$ defined by the f_i. This variety depends on how we present R by generators and relations but if we chose different generators and relations we get an isomorphic variety. We have the following identities

$$\mathbb{V}(\mathbb{C}[\mathbb{X}]) \cong \mathbb{X} \text{ and } \mathbb{C}[\mathbb{V}(R)] \cong R.$$

One can also give a more intrinsic description of $\mathbb{V}(R)$. For every point $p \in \mathbb{V}(R)$ one can look at the embedding $p \hookrightarrow \mathbb{V}(R)$. From the algebraic point of view this will give a map from $R \to \mathbb{C}[p] = \mathbb{C}$, so points correspond to maps from R to $\mathbb{C}$. Such a map is determined by its kernel. As $\mathbb{C}$ is an algebraically closed field these kernels correspond to the maximal ideals of R. So we can also define $\mathbb{V}(R)$ as the set of all maximal ideals of R.

If $f : R \to S$ is a morphism an $\mathfrak{m} \subset S$ is a maximal ideal then $f^{-1}(\mathfrak{m})$ will be a maximal ideal of R, so we have a map

$$\mathbb{V}(S) \to \mathbb{V}(R) : \mathfrak{m} \to f^{-1}(\mathfrak{m}),$$

which is a morphism of affine varieties.

We have only described $\mathbb{V}(R)$ as a set but now we want to give $\mathbb{V}(R)$ some more structure. This can be done by introducing the *Zariski Topology*. This topology can be defined by its closed sets: $C \subset \mathbb{V}(R)$ is closed if there is an ideal $\mathfrak{c} \lhd R$ such that $C = \{\mathfrak{m} \in \mathbb{V}(R) | \mathfrak{c} \subset \mathfrak{m}\}$. Now if $R = \mathbb{C}[x_1, \ldots, x_n]/\mathfrak{i}$ we can see $\mathfrak{c}$ as generated by polynomials (c_i) so the points in $\mathbb{V}(R)$ that lie on C are exactly those for which the c_i are zero. So closed sets are subsets that correspond to zeros of polynomial functions. Every closed set C will give us a morphism $R \to \mathbb{C}[C] \cong R/\mathfrak{c}$ which is a surjection. Conversely every surjection $R \to S$ will give us an embedding of a closed subset $\mathbb{V}(S)$ in $\mathbb{V}(R)$.

Open sets on the other hand are unions of subsets for which certain polynomials are nonzero. Contrarily to closed subsets, open subsets can not always be considered as affine varieties. F.i. in $\mathbb{C}^2$ the complement of the origin is an open subset but it is not isomorphic to an affine variety. Basic open sets are sets on which one polynomial f does not vanish. Such a set can be considered as the variety corresponding to the ring

$$R[1/f] = \left\{ \frac{r}{f^i} | r \in R, i \in \mathbb{N} \right\}.$$

This construction is called a *localization*.

The Zariski topology is not the same as the ordinary complex topology on $\mathbb{V}(R) \subset \mathbb{C}^n$. The ordinary topology has lots more closed (open) sets. For instance a closed ball with finite radius around a point is closed in the ordinary topology of $\mathbb{C}^n$, but not in the Zariski topology because the zeros of a nonzero polynomial never contain closed balls.

The duality between algebras and geometry allows us to make a translation table.

Geometry	Algebra
Affine Variety	Affine Algebra
Morphism	Algebra Morphism
Point	Maximal Ideal
Closed Set	Semiprime Ideal (i.e. $f^n \in \mathfrak{i} \Rightarrow f \in \mathfrak{i}$)
Intersection of closed Sets	Sum of ideals
Union of closed sets	Intersection of ideals
Embedding of a closed subvariety	Surjection
The image is dense	Injection
Connected	does not contain idempotents

A variety is called *irreducible* if it is not the union of two closed subsets. On the algebra level this means that the algebra is a domain: it has no zero divisors. Irreducible varieties have the property that all nonempty open sets are dense. From now on we will work with irreducible varieties.

To an irreducible variety $\mathbb{X} \subset \mathbb{C}^n$ we can associate a dimension. There are several equivalent ways to do this. One of them is to look at the function field

$$\mathbb{C}(\mathbb{X}) := \{\frac{f}{g} | f \in \mathbb{C}[\mathbb{X}], g \in \mathbb{C}[\mathbb{X}] \setminus \{0\}\}.$$

This is a field extension of $\mathbb{C}$ and we can look at its transcendency degree. This is the maximal number of elements you can find in $\mathbb{C}(\mathbb{X})$ that are algebraically independent over $\mathbb{C}$. The *dimension of* $\mathbb{X}$ is by definition equal to the transcendency degree of $\mathbb{C}(\mathbb{X})$. From this definition it is clear that $\mathbb{C}^n$ has dimension n because the transcendency degree of $\mathbb{C}(X_1, \dots, X_n)$ is n. There is also a second characterization of the dimension of $\mathbb{C}[\mathbb{X}]$: it is the length n of the largest chain of prime ideals

$$0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subsetneq R.$$

you can find in R. This notion is also called the *Krull dimension* of R.

1.2 Formal completions

Varieties are a generalization of manifolds to the algebraic setting, but they are a lot more harder to work with. One of the reasons for this is that unlike for manifolds not all points in a variety look the same. Around every point in an n-dimensional manifold we can find an open neighborhood that is diffeomorphic to $\mathbb{R}^n$.

If we want to arrive at something similar in the case of varieties, we must do something more clever than just look at an open neighborhood, because there are very few open subsets in the Zariski topology. To encode the local information of a variety around a point we will need the notion of formal completion.

If R is a ring and $\mathfrak{m}$ an ideal we define the $\mathfrak{m}$-adic completion of R at $\mathfrak{m}$ as the ring

$$\widehat{R}_{\mathfrak{m}} \cong \lim_{\leftarrow} R/\mathfrak{m}^i := \{(x_i)_{i\in\mathbb{N}} | x_i \in R/\mathfrak{m}^{i+1} \text{ and } x_i + \mathfrak{m}^{i+1} = x_j + \mathfrak{m}^{i+1} \text{ if } i < j\}.$$

with componentwise addition and multiplication. If it is clear for which ideal we construct the completion we will use $\widehat{R}$ instead of $\widehat{R}_{\mathfrak{m}}$.

There is a morphism

$$R \to \widehat{R}_{\mathfrak{m}} : r \mapsto (r + \mathfrak{m}^{i+1}),$$

which is an embedding if $\cap_{i=1}^{\infty}\mathfrak{m}^i = 0$. The ring $\widehat{R}_{\mathfrak{m}}$ has an ideal $\widehat{\mathfrak{m}} = \{(x_i)_{i\in\mathbb{N}} | x_i \in \mathfrak{m}/\mathfrak{m}^{i+1}\}$, which is the ideal generated by the image of $\mathfrak{m} \lhd R$ under the natural morphism. It is easy to check that $\widehat{R}/\widehat{\mathfrak{m}}^i \cong R/\mathfrak{m}^i$.

Just like you can approximate a real number by its consecutive decimal approximations, you should see the sequences as consecutive approximations and therefore the ring is called the completion of R. Indeed, if $(r_i) \in \widehat{R}$ we can find $\tilde{r}_i \in \mathfrak{m}^{i-1}$ such that $r_i = \tilde{r}_0 + \cdots + \tilde{r}_i + \mathfrak{m}^i$, so it make sense to write (r_i) as $r = \tilde{r}_0 + \tilde{r}_1 + \tilde{r}_2 + \ldots$. More general it also makes sense to consider infinite sums where the $\tilde{r}_i$ are elements in $\widehat{\mathfrak{m}}^i$ and treat these as elements in $\widehat{R}$.

If you look at the polynomial ring $R = \mathbb{C}[X]$ and the ideal $\mathfrak{m} = (X)$, we can see that the elements in the ring $\widehat{R}_{\mathfrak{m}}$ can be seen as infinite power series. So the ring $\widehat{R}_{\mathfrak{m}}$ is also the ring of formal power series in X. The same holds for more variables

$$\widehat{\mathbb{C}[X_1, \ldots, X_n]}_{(X_1,\ldots,X_n)} = \mathbb{C}[\![X_1, \ldots, X_n]\!].$$

So if you look at functions in $\mathbb{C}[\![X_1, \ldots, X_n]\!]$ you treat them as their 'Taylor' series around the zero. Therefore $\mathbb{C}[\![X_1, \ldots, X_n]\!]$ describes the local geometry around the origin in $\mathbb{C}^n$.

More general if R is an affine ring and $\mathfrak{p}$ is the maximal ideal corresponding to the point $p \in \mathbb{V}(R)$ then $\widehat{R}_{\mathfrak{p}}$ will encode the local information around the point p. This ring will be called the *complete local ring* around p.

Lemma 1.3. *If R is an affine ring over $\mathbb{C}$ and $\mathfrak{p}$ is a maximal ideal then $\widehat{R}_{\mathfrak{p}}$ has a unique maximal ideal, $\widehat{\mathfrak{p}}$.*

Proof. Clearly, there is a morphism $\pi : \widehat{R}_{\mathfrak{p}} \to R/\mathfrak{p} : (u_i) \mapsto u_0$. Because $\mathfrak{p}$ is maximal, $R/\mathfrak{p}$ is $\mathbb{C}$ and hence $\widehat{\mathfrak{p}} = \mathsf{Ker}\pi$ is also maximal.

If $r = (r_i) \notin \mathsf{Ker}\pi$ then we can write it as $c(1-\epsilon)$ where $c = \pi(r) \in \mathbb{C}$ and $\epsilon = 1 - r/c \in \widehat{\mathfrak{p}}$. The formal sum

$$u = c^{-1}(1 + \epsilon + \epsilon^2 + \dots)$$

represents and element in $\widehat{R}_{\mathfrak{p}}$ and $ur = 1$. So $\mathsf{Ker}\pi$ contains all non-invertible elements and hence all proper ideals in $\widehat{R}_{\mathfrak{p}}$. □

Rings with a unique maximal ideal are called *local rings*, so this is the reason we speak of the complete local ring.

Note that the formal completion of an affine algebra is not an affine algebra because it is usually not a finitely generated ring. However, such a ring still has a nice property: it is Noetherian.

Definition 1.4. Let R be a commutative ring and M an R-module. M is called *Noetherian* if every increasing chain of submodules $M_1 \subset M_2 \subset \dots$ in M becomes stationary: $M_i = M_{i+1}$ if $i \gg 0$, or equivalently every submodule of M is finitely generated.

A ring R is *Noetherian* if it is a Noetherian module over itself. In other words every increasing chain of ideals $\mathfrak{m}_1 \subset \mathfrak{m}_2 \subset \dots$ becomes stationary: $\mathfrak{m}_i = \mathfrak{m}_{i+1}$ if $i \gg 0$, or equivalently every ideal is finitely generated.

This is a very important property that is often used in commutative ring theory because it is preserved under many constructions.

Lemma 1.5. *If R is a ring then*

- *every submodule of a Noetherian module is Noetherian*
- *every quotient module of a Noetherian module is Noetherian*
- *if N and M/N are Noetherian then M is Noetherian.*

Proof. The first statement is trivial because every sequence of submodules of $N \subset M$ is a sequence of submodules of M. The second statement follows from the fact that if (U_i) is an increasing sequence of submodules in M/N and $\pi : M \to M/N$ is a projection map then $\pi^{-1}(U_i)$ is an increasing sequence of submodules in M.

Thirdly, if U is a submodule of M then $\pi(U)$ is a submodule of M/N and hence finitely generated by $y_1+N, \dots, y_k+N$, where we can chose $y_i \in U$. If $u \in U$ then we can write $u + N = r_1(y_1 + N) + \dots r_k(y_k + N)$ so $u = r_1y_1 + \dots r_ky_k + n$ with $n \in N \cap U$. Because N is Noetherian, $N \cap U$ is finitely generated by $z_1, \dots, z_l$. So U is generated by $y_1, \dots, y_k, z_1, \dots, z_l$. □

Note that the last statement implies that $M \oplus N$ is Noetherian if both M, N are.

Lemma 1.6. *If R is a Noetherian ring then*

- *$R/\mathfrak{i}$ is Noetherian*
- *$R[X]$ is Noetherian (= Hilbert's basis theorem)*
- *$R[\![X]\!]$ is Noetherian*
- *$\widehat{R}_{\mathfrak{m}}$ is Noetherian.*

Proof. If $R/\mathfrak{i}$ is not Noetherian we have a strictly increasing chain of ideals $\mathfrak{m}_1/\mathfrak{i} \subset \mathfrak{m}_2/\mathfrak{i} \subset \dots$, which gives rise to a strictly increasing chain of ideals $\mathfrak{m}_1 \subset \mathfrak{m}_2 \subset \dots$.

If $\mathfrak{i}$ is an ideal in $R[X]$, we will show that it is finitely generated. Let $f_1, f_2, \dots \subset \mathfrak{i}$ be a sequence of nonzero elements such that f_{i+1} is an element of $\mathfrak{i} \setminus (f_1, \dots, f_i)$ with minimal degree. Let a_i be the highest coefficient of f_i and consider the ideal $\mathfrak{a} = (a_1, a_2, \dots) \lhd R$. Because R is Noetherian there is a $m \in \mathbb{N}$ such that $\mathfrak{a} = (a_1, a_2, \dots, a_m)$. So $a_{m+1} = \sum r_j a_j$ and the element

$$g = f_{m+1} - \sum r_j f_j X^{\deg f_{m+1} - \deg f_j},$$

sits in $\mathfrak{i} - (f_1, \dots, f_m)$ but has a lower degree than f_{m+1}. By the the minimality of the degree of f_{m+1} we have that $g = 0$ and hence $f_{m+1} \in (f_1, f_2, \dots, f_m) = (f_1, f_2, \dots, f_{m+1})$. Continuing like this we see that $\mathfrak{i} = (f_1, f_2, \dots, f_m)$.

The proof of the third statement is completely analogous to the second but now we define $\deg f$ to be the smallest nonzero power of X in $f(X)$.

The fourth statement follows from the first and the third statement because the map $R[\![X_1, \dots, X_m]\!] \mapsto \widehat{R}_{\mathfrak{m}} : f \to (f + \mathfrak{m}^i)$ is surjective. $\square$

Note that a subring of a Noetherian ring is NOT necessarily Noetherian (because a subring is not an ideal). An example of this is $R[X, XY, XY^2, \dots] \subset R[X, Y]$.

Lemma 1.7. *If R is a Noetherian ring then the Noetherian modules are precisely the finitely generated modules.*

Proof. First remark that by definition every Noetherian module is finitely generated. A module M is finitely generated if it is a quotient $R^{\oplus k}$. If R is Noetherian then $R^{\oplus k}$ is Noetherian so M is also Noetherian. $\square$

Given an R-module M we can also construct $\widehat{M}_{\mathfrak{m}}$:

$$\widehat{M}_{\mathfrak{m}} \cong \lim_{\leftarrow} M/\mathfrak{m}^i M := \{(x_i)_{i\in\mathbb{N}} | x_i \in M/\mathfrak{m}^i M \text{ and } x_i + \mathfrak{m}^i M = x_j + \mathfrak{m}^i M \text{ if } i < j\},$$

which has the structure of an $\widehat{R}_{\mathfrak{m}}$-module by componentwise multiplication

$$(r_i)_{i\in\mathbb{N}}(x_i)_{i\in\mathbb{N}} = (r_i x_i)_{i\in\mathbb{N}}.$$

If $\phi : M \to N$ is a morphism of R-modules (i.e. $\phi(rm) = r\phi(m)$) then there is also a morphism $\widehat{\phi} : \widehat{M} \to \widehat{N} : (x_i)_{i\in\mathbb{N}} \mapsto (\phi(x_i))_{i\in\mathbb{N}}$. This means that there is a functor

$$\widehat{-} : \texttt{Mod} - R \to \texttt{Mod} - \widehat{R} : M \to \widehat{M}.$$

Where $\texttt{Mod} - R$ stands for the category of all R-modules.

Definition 1.8. A sequence of modules $M_1 \xrightarrow{\phi} M_2 \xrightarrow{\psi} M_3$ is called exact if $\mathsf{Ker}\psi = \mathsf{Im}\phi$.

A functor is called exact if it maps exact sequences to exact sequences.

Lemma 1.9. *If $M_1 \xrightarrow{\phi} M_2 \xrightarrow{\psi} M_3$ is exact and M_1, M_2, M_3 are Noetherian then $\widehat{M_1} \xrightarrow{\phi} \widehat{M_2} \xrightarrow{\psi} \widehat{M_3}$ is exact.*

Proof. For the proof we refer to `http://pub.math.leidenuniv.nl/~bieselod/teaching/2013-2014/CommAlg/Lecture8.pdf` □

Corollary 1.10. *If $R = \mathbb{C}[X_1, \dots, X_n]/(f_1, \dots, f_k)$ and $\mathfrak{m} = (X_1, \dots, X_n) \supset (f_1, \dots, f_k) = \mathfrak{f}$ then*

$$\widehat{R}_{\mathfrak{m}/\mathfrak{f}} = \mathbb{C}[\![X_1, \dots, X_n]\!]/(f_1, \dots, f_k)$$

Proof. We have the following exact sequence of $\mathbb{C}[X_1, \dots, X_n]$-modules

$$\mathbb{C}[X_1, \dots, X_n]^{\oplus k} \xrightarrow{\cdot f_i} \mathbb{C}[X_1, \dots, X_n] \to R.$$

After completion we get

$$\mathbb{C}[\![X_1, \dots, X_n]\!]^{\oplus k} \xrightarrow{\cdot f_i} \mathbb{C}[\![X_1, \dots, X_n]\!] \to \widehat{R}_{\mathfrak{m}/\mathfrak{f}}.$$

This is again exact because the $\mathbb{C}[X_1, \dots, X_n]$-modules are Noetherian. Therefore the kernel of $\mathbb{C}[\![X_1, \dots, X_n]\!] \to \widehat{R}_{\mathfrak{m}/\mathfrak{f}}$ is generated by the f_i. □

Given a complete local ring we can write it as $\mathbb{C}[\![X_1, \dots, X_n]\!]/(f_1, \dots, f_k)$, but it is possible to simplify this description using automorphisms of the ring $\mathbb{C}[\![X_1, \dots, X_n]\!]$.

Lemma 1.11. *For every* $Y_1, \dots, Y_n \in \widehat{\mathfrak{m}} = (X_1, \dots, X_n) \lhd \mathbb{C}[\![X_1, \dots, X_n]\!]$ *there is morphism* ϕ *of* $\mathbb{C}[\![X_1, \dots, X_n]\!]$ *that maps* $X_i \mapsto Y_i$. *This morphism is an automorphism if and only if*

$$\{Y_1 + \widehat{\mathfrak{m}}^2, \dots, Y_n + \widehat{\mathfrak{m}}^2\}$$

form a basis for $\widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2$.

Proof. If $r = (r_i(X_1, \dots, X_n)) \in \lim_{\leftarrow} \mathbb{C}[X_1, \dots, X_n]/\widehat{\mathfrak{m}}^i$, we define

$$\phi(r) = (r_i(Y_1, \dots, Y_n) + \widehat{\mathfrak{m}}^i).$$

It is easy to check that this is well defined and a ring morphism. If this morphism is an isomorphism then $\phi(\widehat{\mathfrak{m}}) = \widehat{\mathfrak{m}}$ because $\widehat{\mathfrak{m}}$ is the unique maximal ideal in $\mathbb{C}[\![X_1, \dots, X_n]\!]$. This implies that the Y_i must induce a basis for $\widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2$.

Now suppose that Y_i induce a basis then there is an $M \in \mathsf{GL}_n(\mathbb{C})$ such that $\sum M_{ij} Y_j = X_i \mod \widehat{\mathfrak{m}}^2$. This M induces a morphism

$$\phi_M : \mathbb{C}[\![X_1, \dots, X_n]\!] \to \mathbb{C}[\![X_1, \dots, X_n]\!] : X_i \to \sum_j M_{ij} X_j$$

and $\psi := \phi \circ \phi_M$ is a morphism for which $\psi(X_i) = X_i \mod \widehat{\mathfrak{m}}^2$. If we can prove that ψ is an automorphism then both ϕ and ϕ_M are also automorphisms.

Set $\epsilon : \mathbb{C}[\![X_1, \dots, X_n]\!] \to \mathbb{C}[\![X_1, \dots, X_n]\!] : u \mapsto u - \psi(u)$. If the minimal degree of u is n then the minimal degree of $\epsilon(u)$ is bigger than n and therefore the sum

$$\chi(u) := u + \epsilon(u) + \epsilon \circ \epsilon(u) + \epsilon \circ \epsilon \circ \epsilon(u) + \dots$$

is convergent in $\mathbb{C}[\![X_1, \dots, X_n]\!]$ and $\chi \circ \psi(u) = u$. □

Corollary 1.12. *Given a complete local ring we can write it as*

$$\mathbb{C}[\![X_1, \dots, X_n]\!]/(f_1, \dots, f_k),$$

where $(f_1, \dots, f_k) \subset (X_1, \dots, X_n)^2$.

Proof. If f_i contains a linear term we can find an automorphism such that f_i becomes a variable and then we can delete it from the generators. □

The easiest complete local ring is the ring of formal power series $\mathbb{C}[\![X_1, \dots, X_m]\!]$ and therefore we introduce the following definition.

Definition 1.13. We will call a point $p \in \mathbb{X}$ *smooth* if the complete local ring $\widehat{\mathbb{C}[\mathbb{X}]_{\mathfrak{p}}}$ is isomorphic to a ring of formal power series $\mathbb{C}[\![X_1, \dots, X_m]\!]$.

If this is not the case then we call the point a *singular point* or *singularity.* Two singular points are called *equivalent* if their complete local rings are isomorphic.

Singularity theory studies singularities up to this equivalence.

A simple test to check whether two singularities non-isomorphic complete local rings is to compare their Hilbert series. For a local ring $\widehat{R}$ with maximal ideal $\mathfrak{m}$ we define the *Hilbert series*

$$h_i(\widehat{R}) = \dim \mathfrak{m}^i/\mathfrak{m}^{i+1}.$$

Clearly if $\widehat{R} \cong \widehat{S}$ then both have a unique maximal ideal and their hibert series will be the same. The converse is not always true: non-isomorphic complete local rings can have the same Hilbert series. In some cases however the Hilbert series is enough to determine the complete local ring.

Lemma 1.14. *The Hilbert series of* $\widehat{R} = \mathbb{C}[\![X_1, \dots, X_n]\!]$ *is*

$$h_i(\widehat{R}) = \binom{n+i-1}{n-1}.$$

If a complete local ring $\mathbb{C}[\mathbb{X}]_{\mathfrak{p}}$ *has the same Hilbert series as* $\mathbb{C}[\![X_1, \dots, X_n]\!]$ *then it is isomorphic to* $\mathbb{C}[\![X_1, \dots, X_n]\!]$*.*

Proof. h_i just calculates the number of monomials of degree i in the polynomial ring, which is an i-combination with repetitions out of n elements.

To go in the opposite direction assume $\widehat{R}$ is complete local with maximal ideal $\mathfrak{m}$ such that $h_1(\widehat{R}) = n$. Choose elements $Y_1, \dots, Y_n$ in $\mathfrak{m}$ such that the $Y_i + \mathfrak{m}$ form a basis for $\mathfrak{m}/m^2$. Note that in this case the $Y_i + \mathfrak{m}^i$ generate $\widehat{R}/\mathfrak{m}^i$ (this can easily be proved by induction).

We can define a morphism of complete rings

$$\mathbb{C}[\![X_1, \dots, X_n]\!] \to \widehat{R} : (f_i)_{i \in \mathbb{N}} \mapsto (f_i(Y_1 + \mathfrak{m}^i, \dots, Y_n + \mathfrak{m}^i))_{i \in \mathbb{N}}$$

which is surjective mod $\mathfrak{m}^i$. Because the Hilbert series are the same

$$\dim \mathbb{C}[\![X_1, \dots, X_n]\!]/(X)^i = \dim \widehat{R}/\mathfrak{m}^i,$$

so the map is also injective mod $(X)^i$. Therefore it is bijective. $\square$

1.3 Tangent spaces and smooth points

Given a point p in a variety $\mathbb{X}$, the local ring $\widehat{\mathbb{C}[\mathbb{X}]_{\mathfrak{p}}}$ can be thought of as the ring of all power series around p. If we have any function in $f \in \widehat{\mathbb{C}[\mathbb{X}]_{\mathfrak{p}}}$ its n^{th} order Taylor approximation can be seen as $f \mod \mathfrak{p}^{n+1}$. In particular the linear part of the function is an element in $\mathfrak{p}/\mathfrak{p}^2$. In differential geometry the linear part of the Taylor expansion is an element of the cotangent space so therefore it makes sense to define

Definition 1.15. If $\mathbb{X}$ is an affine variety and $p \in \mathbb{X}$ is a point with maximal ideal $\mathfrak{p} \lhd \mathbb{C}[\mathbb{X}]$ then we define the *cotangent space* and *tangent space* at p as

$$T_p^*\mathbb{X} = \frac{\mathfrak{p}}{\mathfrak{p}^2} \text{ and } T_p\mathbb{X} = (\frac{\mathfrak{p}}{\mathfrak{p}^2})^*.$$

(Note that $-^*$ stands for the dual vector space).

If $\phi : \mathbb{X} \to \mathbb{Y}$ is a morphism that maps p to q we have a corresponding map $\phi^* : \mathbb{C}[\mathbb{Y}] \to \mathbb{C}[\mathbb{X}]$ with $\phi^{*-1}(\mathfrak{p}) = \mathfrak{q}$. Because $\phi^*(\mathfrak{q}^i) \subset \mathfrak{p}^i$, this also gives a map $d\phi^* : \mathfrak{q}/\mathfrak{q}^2 \to \mathfrak{p}/\mathfrak{p}^2$.

Let us look at the case where $\mathbb{Y} = \mathbb{C}^n$, $\mathbb{C}[\mathbb{X}] = \mathbb{C}[X_1, \dots, X_n]/(f_1, \dots, f_m)$ and $\mathfrak{q} = (X_1, \dots, X_n)$. We can identify $\mathfrak{q}/\mathfrak{q}^2$ with the vector space $\mathbb{C}^n$ by using the basis $dX_i := X_i + \mathfrak{q}^2$. The inclusion $\mathbb{X} \subset \mathbb{Y}$ gives a surjective map $\phi^* : \mathbb{C}[X_1, \dots, X_n] \to \mathbb{C}[X_1, \dots, X_n]/(f_1, \dots, f_m)$. Therefore $d\phi^*$ will also be surjective and its kernel will be all those $\sum_i \alpha_i dX_i$ such that

$$\sum_i \alpha_i X_i + \mathfrak{q}^2 \in \phi^{*-1}\mathfrak{p}^2 = \mathfrak{q}^2 + (f_1, \dots, f_n).$$

Because $\mathfrak{q}^2 + (f_1, \dots, f_n)$ is generated by all degree 2 monomials and the linear parts of the f_j, the kernel of the map $d\phi^*$ is generated by all

$$\sum_i \frac{df_j}{dX_i}(0) dX_i.$$

For the dual space $(\mathfrak{q}/\mathfrak{q}^2)^*$ we have the dual basis ∂_i (such that $\partial_i(dX_j) = \delta_{ij}$). The vectors in $(\mathfrak{p}/\mathfrak{p}^2)^*$ are those linear combinations $\sum_i \beta_i \partial_i$ that evaluate zero on $\mathsf{Ker}\, d\phi^*$ so

$$\mathrm{T}_p\mathbb{X} \cong \{\beta \in \mathbb{C}^n | \sum_i \frac{\partial f_j}{\partial x_i}(p)\beta_i = 0,\ j = 1, \dots, k\}.$$

This corresponds to the classical definition of the tangent space of an embedded variety.

Now look at the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

The tangent space at p is the kernel of this matrix evaluated at p, so if the tangent space in p is k-dimensional then the rank of this matrix is $n-k$ and there is an $n-k \times n-k$-minor that is nonzero. This means that in an open neighborhood of p this minor is also nonzero and hence the tangent space in all these points is at most k-dimensional. The points for which the tangent space has the lowest dimension form an Zariski-open subset in $\mathtt{Smooth} \subset \mathbb{X}$. This subset is dense if $\mathbb{X}$ is irreducible. For each point in $\mathtt{Smooth}$ there is a $n-k \times n-k$-minor with nonzero determinant.

Theorem 1.16. *Let $\mathbb{X}$ be an irreducible variety and p a point in $\mathbb{X}$ with maximal ideal $\mathfrak{p} \lhd \mathbb{C}[\mathbb{X}]$. Then $\widehat{\mathbb{C}[\mathbb{X}]}_{\mathfrak{p}}$ is isomorphic to $\mathbb{C}[\![X_1, \dots, X_k]\!]$ if and only if the dimension of the tangent space $\dim \mathrm{T}_p\mathbb{X} = k$ is minimal.*

Proof. Without loss of generality we can assume the point p to be the zero point in $\mathbb{X} \subset \mathbb{C}^n$.

We only show the proof for hypersurfaces. In that case $\mathbb{C}[\mathbb{X}] = \mathbb{C}[X_1, \dots, X_n]/(f)$ where f is an irreducble polynomial without constant term. The Jacobian matrix in the zero point has either rank one (if f has a linear term) or rank zero (if f has no linear term).

If f has a linear term, we can do a linear base change in the variables such that the linear term of f is X_n. By lemma 1.11 there is an automorphism ϕ of $\mathbb{C}[\![X_1, \dots, X_n]\!]$ such that $\phi(X_i) = X_i$ if $i < n$ and $\phi(X_n) = f$. Therefore

$$\mathbb{C}[\![X_1, \dots, X_n]\!]/(f) \cong \mathbb{C}[\![X_1, \dots, X_n]\!]/(X_n) \cong \mathbb{C}[\![X_1, \dots, X_{n-1}]\!].$$

To go in the opposite direction, suppose that $\widehat{R} = \mathbb{C}[X_1, \dots, X_n]/(f)$. If f has no linear term then $h_1(\widehat{R}) = n$. So if $\widehat{R}$ is a formal power series ring, it should be a formal power series ring in n variables. But if $f \in (X_1, \dots, X_n)^i \setminus (X_1, \dots, X_n)^{i+1}$ then, for the same reason as in the proof of lemma 1.14, $h_i(\widehat{R}) < h_i(\mathbb{C}[\![X_1, \dots, X_n]\!])$. □

2
Hypersurface singularities

2.1 Hypersurfaces and critical points

Recall that a *hypersurface* is the zero locus of a single polynomial and hence it corresponds to an affine ring of the form $\mathbb{C}[X_1, \ldots, X_n]/(f)$. Note that a hypersurface is an irreducible variety if f is an irreducible polynomial.

Given a function $f \in \mathbb{C}[X_1, \ldots, X_n]$, we can define a 1-parameter family of hypersurfaces $\mathbb{X}_\lambda$ defined by the equations $f(X) - \lambda = 0$.

Lemma 2.1. *The smooth points of $\mathbb{X}_\lambda$ are those points $(\xi_1, \ldots, \xi_n)$ such that $f(\xi) = \lambda$ and $\nabla f(\xi) := (\frac{\partial f(\xi)}{\partial X_1}, \ldots, \frac{\partial f(\xi)}{\partial X_n}) \neq 0$.*

Proof. This follows straight from the previous chapter. □

If $\xi \in \mathbb{C}^n$ is a point for which $\nabla f = 0$ then ξ will be a singular point of the variety $\mathbb{X}_{f(\xi)}$.

Definition 2.2. A *critical point* of $f \in \mathbb{C}[X_1, \ldots, X_n]$ is a point ξ such that $\nabla f(\xi) = 0$.

The *Hessian* of a critical point ξ is the matrix

$$H_{ij} = \left(\frac{\partial^2 f}{\partial X_i \partial X_j}\right)_\xi$$

The Hessian defines a bilinear form on the tangent space. If we perform a base change for the coordinates we get

$$H'_{ij} = \sum_{k,l} H_{kl} \frac{\partial x_k}{\partial y_i} \frac{\partial x_l}{\partial y_j}.$$

Standard linear algebra shows that we can bring this matrix into a diagonal matrix with $0, 1's$ on the diagonal. (This is Sylvester's Law of inertia over the complex numbers, if we work over the real numbers this matrix also can have -1 on the diagonal.) For the function f this implies that after a base change we can bring f in the form

$$X_1^2 + \cdots + X_k^2 + r(X) \text{ with } r(X) \in \mathfrak{m}^3$$

The number k is the rank of the Hessian.

Definition 2.3. If $f \in \mathbb{C}[X_1, \ldots, X_n]$ and ξ is a critical point then the *corank* of ξ is $c := n - k$ where k is the rank of the Hessian. A critical point with corank 0 is called *non-degenerate* and if the corank is nonzero it is called *degenerate*.

Theorem 2.4. *All nondegenerate critical points with the same rank define equivalent singularities.*

Proof. We need to show that all rings of the form

$$\mathbb{C}[\![X_1, \ldots, X_n]\!]/(X_1^2 + \cdots + X_n^2 + g)$$

with $g \in (X_1, \ldots, X_n)^3$ are isomorphic.

The idea is to perform a change of coordinates $X_i = Y_i + l_i$ with $l_i \in (X_1, \ldots, X_n)^2$ such that the new g becomes zero. We do this step by step. Suppose we have found $l_i^j \mod (X_1, \ldots, X_n)^j$ such that $g^j = 0 \mod (X_1, \ldots, X_n)^j = 0$. Then the new $g^{j+1} \mod (X_1, \ldots, X_n)^{j+1}$ becomes

$$\sum_i 2Y_i l_i^{j+1} + g^j(Y) \mod (X_1, \ldots, X_n)^{j+1}.$$

Because all the Y_i are present we can easily find l_i^{j+1} to cancel the j-degree term of g^j. □

Remark 2.5. A similar proof shows that if the corank is c then we can perform a transformation such that the local ring looks like

$$\mathbb{C}[\![X_1, \ldots, X_n]\!]/(X_{c+1}^2 + \cdots + X_n^2 + g)$$

with $g \in (X_1, \ldots, X_c)^3 \subset \mathbb{C}[\![X_1, \ldots, X_c]\!]$.

Definition 2.6. A function $f \in \mathbb{C}[X_1, \ldots, X_n]$ is called a *Morse function* if it has only nondegenerate critical points and all critical points have a different value for f.

2.2 Versal deformations

If $\mathbb{C}[X_1, \ldots, X_n]/(f)$ has a singularity at the origin, we can look at the deformation theory of this singularity.

Definition 2.7. A formal deformation of $f \in \mathbb{C}[\![X_1, \ldots, X_n]\!]$ with base $\Lambda = \mathbb{C}^\ell$ is a function $F \in \mathbb{C}[\![X_1, \ldots, X_n, \lambda_1, \ldots, \lambda_\ell]\!]$ such that

$$F(x, 0) =: F \mod \langle \lambda_1, \ldots, \lambda_\ell \rangle = f(x).$$

We will say that F is a polynomial/holomorphic deformation if F is the formal power series of a polynomial/holomorphic map $F : \mathbb{C}^n \times \Lambda \to \mathbb{C}$.

Note that two functions $f_1, f_2 \in \mathbb{C}[\![X_1, \ldots, X_n]\!]$ are considered equivalent if there is an automorphism $\phi \in \mathsf{Aut}(\mathbb{C}[\![X_1, \ldots, X_n]\!])$ such that $\phi(f_1) = f_2$. Such an automorphism can also be interpreted as a base change: define $g_i = \phi(X_i)$ then $\phi(f_1) = f_1(g_1, \ldots, g_n)$.

Keeping this in mind we say:

Definition 2.8. Two formal deformations F_1, F_2 are *equivalent* if there are elements

$$g_i \in \mathbb{C}[\![X_1, \ldots, X_n, \lambda_1, \ldots, \lambda_\ell]\!]$$

such that $g_i \mod \langle \lambda_1, \ldots, \lambda_\ell \rangle = X_i$ and

$$F_1(g_1, \ldots, g_n, \lambda_1, \ldots, \lambda_\ell) = F_2 \in \mathbb{C}[\![X_1, \ldots, X_n, \lambda_1, \ldots, \lambda_\ell]\!].$$

Given a formal deformation F we can construct new deformations by a change of base space.

Definition 2.9. Let $u_1, \ldots, u_\ell \in \mathbb{C}[\![\kappa_1, \ldots, \kappa_k]\!]$ then the *induced deformation* F' with parameter space $K = \mathbb{C}^k$ is

$$F' = F(X_1, \ldots, X_n, u_1, \ldots, u_l) \in \mathbb{C}[\![X_1, \ldots, X_n]\!] \otimes \mathbb{C}[\![\kappa_1, \ldots, \kappa_k]\!].$$

A deformation F is called *versal* if every deformation is equivalent to a deformation induced from F. If Λ has the smallest possible dimension of all versal deformations then the deformation is called *miniversal.*

Example 2.10. $F(x, \lambda) = x^2 + \lambda$ is a versal deformation of x^2 because every deformation must look like

$$G(x, \mu) = \alpha(x, \mu)x^2 + \beta(\mu)x + \gamma(\mu)$$

with $\alpha(x,0)=1, \beta(0)=\gamma(0)=0$. Because we are working with power series we can construct $\alpha^{-1/2}$ and if we put $g(x)=\alpha^{-1/2}x-\beta(\mu)/2\alpha^{-1/2}$ we get that

$$G'(x,\mu)=G(g,\mu)=x^2+h(\mu)$$

for some function $h(\mu)$. Because $\ell=1$ the deformation is also miniversal.

This example can be generalized to arbitrary hypersurface singularities.

2.3 The Jacobi algebra

Definition 2.11. To a critical point $\xi=0$ of f we can associate its *Jacobi algebra.*

$$\mathtt{Jac}(f)=\mathbb{C}[\![X_1,\ldots,X_n]\!]/(\frac{\partial f}{\partial X_1},\ldots,\frac{\partial f}{\partial X_n})$$

What is the interpretation of the Jacobi algebra? On the ring $\mathbb{C}[\![X_1,\ldots,X_n]\!]$ we have an action of $\mathsf{G}=\mathsf{Aut}\mathbb{C}[\![X_1,\ldots,X_n]\!]$ and functions that are in the same G-orbit are considered equivalent. The tangent space space to the orbit $G\cdot f$ is $\mathfrak{g}\cdot f$, where $\mathfrak{g}$ is the Lie algebra of G. The Lie algebra of the group of automorphism is the space of derivations because if $(1+\epsilon\psi)$ is an automorphism then

$$(1+\epsilon\psi)(ab)=(1+\epsilon\psi)(a)(1+\epsilon\psi)(b)=ab+\epsilon(\psi(a)b+a\psi(b))\Rightarrow\psi(ab)=\psi(a)b+a\psi(b)$$

so ψ is a derivation. Every derivation is characterized by the images $\psi(X_i)$ and one can calculate that

$$\psi(f)=\sum_i\psi(X_i)\frac{\partial f}{\partial X_i}.$$

These are precisely the elements of the Jacobian ideal, so the Jacobi algebra can be seen as the normal space to the orbit $G\cdot f\subset\mathbb{C}[\![X_1,\ldots,X_n]\!]$.

Lemma 2.12. *A critical point is nondegenerate if and only if* $\mathtt{Jac}(f)\cong\mathbb{C}$.

Proof. If the critical point is nondegenerate we can assume that $f=X_1^2+\cdots+X_n^2$ and the Jacobi algebra is

$$\mathbb{C}[\![X_1,\ldots,X_n]\!]/(2X_1,\ldots,2X_n)=\mathbb{C}$$

The reverse also holds if $\mathtt{Jac}(f)=\mathbb{C}$ then the zero is a nondegenerate critical point because then $(\frac{\partial f}{\partial X_1},\ldots,\frac{\partial f}{\partial X_n})=(X_1,\ldots X_n)$. This implies that the linear terms of $\frac{\partial f}{\partial X_j}$ span $\langle X_1,\ldots,X_n\rangle$, so the Hessian is nondegenerate. $\square$

Theorem 2.13. *Let $f \in \mathbb{C}[X_1, \dots, X_n]$ then 0 is an isolated critical point if and only if the Jacobi algebra is finite dimensional.*

Proof. If $f \in \mathbb{C}[X_1, \dots, X_n]$ then the critical points of f in $\mathbb{C}^n$ are defined by the equations

$$\frac{\partial f}{\partial X_1} = 0, \dots, \frac{\partial f}{\partial X_n} = 0.$$

Therefore

$$\mathbb{C}[\mathtt{Crit}_f] = \mathbb{C}[X_1, \dots, X_n] / \sqrt{(\frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n})}.$$

0 is an isolated point of $\mathbb{C}[\mathtt{Crit}_f]$ if and only if the completion of this ring at $(X_1, \dots, X_n)$ is isomorphic to $\mathbb{C}$.

This means that

$$\mathbb{C}[\![X_1, \dots, X_n]\!] / \sqrt{(\frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n})} = \mathbb{C}$$

or equivalently if we divide out all nilpotent elements in

$$\mathtt{Jac}(f) = \mathbb{C}[\![X_1, \dots, X_n]\!] / (\frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n})$$

we get $\mathbb{C}$. The latter is true if and only if this algebra is finite dimensional. Indeed, suppose that the maximal ideal contains only nilpotent elements then $X_i^{j_i}$ is zero in $\mathtt{Jac}(f)$, so all monomials of degree at least $j_1 + \dots + j_n$ must be zero because at least one of the X_i must have a power bigger than j_i.

On the other hand if $\mathtt{Jac}(f)$ is finite dimensional then $1, X_j, \dots, X_j^r$ must be linearly dependent for some r. If $g(X_j) = 0$ and g has 2 nonzero terms then bringing in front the lowest common power of X_j we get $X_j^k(1 + \dots) = 0$. Because $(1 + \dots)$ is invertible in $\mathbb{C}[\![X_1, \dots, X_n]\!]$, we have that $X_j^k = 0$. So all generators are nilpotent and hence $(X_1, \dots, X_n)$ contains only nilpotent elements. □

Remark 2.14. In the previous theorem we studied the critical points of f, but we can also study the singularities of $f^{-1}(0)$, which are the critical points of f with $f = 0$. To study these we can study a similar algebra: the Tjurina algebra

$$\mathtt{Tjr}(f) = \mathbb{C}[\![X_1, \dots, X_n]\!] / (f, \frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n}).$$

This algebra has the property that if $f \in \mathbb{C}[X_1, \dots, X_n]$ such that $f(0) = 0$ then 0 is an isolated singularity if and only if the Tjurina algebra is finite dimensional.

The Jacobi algebra can also be used to construct a versal deformation.

Theorem 2.15. *Let f be a function and choose functions $\phi_1, \dots, \phi_\mu \in \mathbb{C}[\![X_1, \dots, X_n]\!]$ such that $\bar{\phi}_1, \dots, \bar{\phi}_\mu$ form a basis for $\mathtt{Jac}(f)$ then*

$$f + \lambda_1 \phi_1 + \cdots + \lambda_\mu \phi_\mu$$

is a versal deformation of f.

Proof. We only sketch the main idea, which is the following. The space of all functions is $\mathbb{C}[\![X_1, \dots, X_n]\!]$ and on this space we have an action of $\mathsf{G} = \mathsf{Aut}\mathbb{C}[\![X_1, \dots, X_n]\!]$. A deformation corresponds to a map $\Lambda \to \mathbb{C}[\![X_1, \dots, X_n]\!]$ which maps the zero to f. A deformation can be induced from another deformation if near f the orbits hit by the former are also hit by the latter, so a deformation is versal if it hits all orbits near f. The image of Λ can also be chosen transversal to the orbit of f because directions that stay in the orbit of f are unnecessary. The Jacobi algebra can be seen as the normal space to the orbit $G \cdot f$. If we choose representatives for the basis elements of $\mathtt{Jac}$ we can map $\mathtt{Jac}$ to $\mathbb{C}[\![X_1, \dots, X_n]\!]$ and the image will be transversal to $G \cdot f$.

Further details can be found in [**?**]. □

2.4 Multiplicity and modality

Definition 2.16. We call $\mu := \dim_{\mathbb{C}} \mathtt{Jac}(f)$ the *multiplicity or Milnor number* of the critical point.

The name multiplicity comes from the following lemma. If we look at a deformation $f + \lambda g$ where $g = s_1 X_1 + \cdots + s_n X_n$ is linear, the critical points of $f + \lambda g$ are solutions to

$$\frac{\partial f}{\partial X_i} - \lambda s_i.$$

We can express the locations of these critical points in terms of λ. If we let λ approach zero then some of these critical points will move to the zero and become one critical point for f. If we reverse the film the critical point of f breaks up in a number of critical points of $f + \lambda g$.

Lemma 2.17 (Milnor)**.** *If $\mu < \infty$ then for almost all $(s_1, \dots, s_n) \in \mathbb{C}^n$ the critical point for f at $x = 0$ breaks up in μ critical points for $f + \lambda g$. These new critical points are nondegenerate.*

Example 2.18. Consider $f(X, Y) = X^3 + Y^3$ then the multiplicity is

$$\dim_{\mathbb{C}} \mathbb{C}[X, Y]/(3X^2, 3Y^2) = 4.$$

If we deform $f(X,Y) = X^3 + sX + Y^3 + tY$ we see that there are 4 critical points $(\pm\sqrt{-s/3}, \pm\sqrt{-t/3})$.

Corollary 2.19. *If a singular point with multiplicity μ breaks down in smaller singularities, the sum of the multiplicities of the smaller singularities is equal to μ*

Now look at a versal deformation $f+g = f+\lambda_1 g_1 + \cdots + \lambda_\mu g_\mu$ of f coming from the Jacobi algebra. If we deform the function several things can happen. If g has a constant term then $f+g$ will become invertible and the ring $\mathbb{C}[\![X_1, \ldots, X_n]\!]/(f+g)$ will be zero. If this happens the deformed hypersurface moves away from the zero.

A second thing that can happen is that 0 stays on the surface but it becomes a smooth point. For this to happen, g should have no constant term but a nonzero linear term. These form a $\mu - 1$-dimensional subspace of $\mathtt{Jac}(f)$.

The last case happens when 0 stays a singularity. Then both the constant and the linear terms have to be zero. If the rank of the Hessian is k then k of the generators of the Jacobian ideal will have a linear term. So using the relations we can already make k of the n linear terms in g zero. To make the others zero we have to impose $c = n - k$ conditions so the g which have a vanishing linear term will form a $\mu - c - 1$-dimensional subspace of $\mathtt{Jac}(f)$.

This $\mu - c - 1$-dimensional space can further be broken down in smaller spaces, an open part of it will contain the deformations that have a nondegenerate singularity the zero (those are the ones for which the quadratic terms do not vanish. Then we can look at the strata containing singularities with multiplicities $2, 3, \ldots$. The highest multiplicity that can occur is that of the original singularity itself because of corollary 2.19.

Definition 2.20. The modality of a critical point $\xi = 0$ of f is the dimension of the stratum in the miniversal deformation for which f_λ has the same multiplicity as f:

$$m(f) = \dim\{\lambda | \mu(f_\lambda) = \mu(f)\}$$

Example 2.21. We will determine the modality of $X^2 + Y^n$. The Jacobi algebra is

$$\mathtt{Jac}(X^2 + Y^n) = \mathbb{C}[\![X,Y]\!]/(2X, nY^{n-1})$$

which has as a basis $1, Y, \ldots, Y^{n-2}$, so $\mu = n - 1$. The Jacobi algebra of

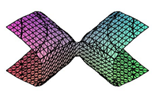
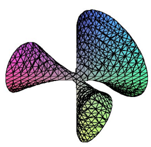
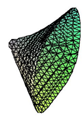
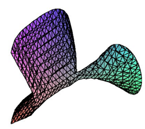
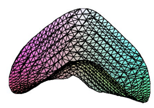
$$\begin{aligned}&\mathtt{Jac}(X^2 + Y^n + a_{n-2}Y^{n-2} + \cdots + a_0)\\ &\quad = \mathbb{C}[\![X,Y]\!]/(2X, nY^{n-1} + (n-2)a_{n-2}Y^{n-3} + \ldots)\\ &\quad = \mathbb{C}[\![X,Y]\!]/(2X, a_i Y^{i-1})\end{aligned}$$

where i is the smallest number such that $a_i \neq 0$. So $\mu(X^2 + Y^n + a_{n-2}Y^{n-2} + \cdots + a_0) = n - 1$ if and only if all a_i are zero. Therefore the modality is 0.

The modality of a critical point describes the number of parameters of the family this critical point is part of. It is an important invariant because it is possible to classify singularity with a small modality. Singularities with modality 0,1,2 are called *simple*, *unimodal* and *bimodal*.

We end this chapter with an important result: the classification of the simple singularities.

Theorem 2.22 (Arnol'd)**.** *Let $\widehat{R}$ be the local ring of a simple hypersurface singularity of dimension 2 then $\widehat{R}$ is isomorphic to $\mathbb{C}[\![X,Y,Z]\!]/(f)$ with f equal to one of the following polynomials*

$A_n,\ n \geq 1$	$D_n,\ n \geq 4$	E_6	E_7	E_8
$XY - Z^{n+1}$	$X^2 + Y^2Z + Z^{n-1}$	$X^2 + Y^3 + Z^4$	$X^2 + Y^3 + YZ^3$	$X^2 + Y^3 + Z^5$
				

These singulaties are called the Kleinian singularities *or* Du Val singularities. *In higher dimensions the classification is the same only now f has extra terms of the form X_i^2 for the extra variables.*

Remark 2.23. For D_n we assumed that $n \geq 4$ because D_1 gives the zero ring, D_2 gives $Z = -X^2(1+Y^2)^{-1}$ so the ring $\widehat{\widehat{R}} = \mathbb{C}[\![X,Y]\!]$ is smooth, and finally D_3 is the same singularity as A_3 (perform the change of variables $X' = X, Y' = Y/\sqrt{2}, Z' = Z + Y^2/2$).

The proof of this theorem is beyond the scope of these notes. It can be found in the book [**?**]. In the following chapters we will study these singularities in detail and see how this classification fits in the bigger picture of modern mathematics.

3

Quotient singularities

3.1 Quotients and rings of invariants

Suppose we have a finite group G and let V be a finite dimensional representation with dimension k. This gives a map $\rho_V : \mathsf{G} \to \mathsf{GL}(V)$ and we write $g \cdot v$ for $\rho_V(g)v$. For every point $v \in V$ we can define the orbit $\mathsf{G} \cdot v := \{g \cdot v | g \in \mathsf{G}\}$. Orbits never intersect so we can partition V into its orbits. We will denote the set of all orbits by V/G.

Because V itself is an affine variety, a natural question one can ask is whether the set V/G can also be given the structure of an affine variety. In the case of finite groups this will be possible, but for general groups there will be extra complications.

We can take a closer look at the problem by looking at the algebraic side of the story. The ring of polynomial functions over V is $R = \mathbb{C}[V] \cong \mathbb{C}[X_1, \dots, X_k]$ is a graded polynomial ring if we give the X_i degree 1.

On R we have an action of G:

$$\mathsf{G} \times \mathbb{C}[V] \to \mathbb{C}[V] : (g, f) \mapsto g \cdot f := f \circ \rho_V(g^{-1}).$$

This action is linear and compatible with the algebra structure: $g \cdot f_1 f_2 = (g \cdot f_1)(g \cdot f_2)$. As $g \cdot X_i : \sum_j \rho_V(g^{-1})_{ij} X_j$ is homogeneous of degree 1 the G-action maps homogeneous elements to homogeneous elements with the same degree.

The ring R contains a subring consisting of the invariant functions

$$S := R^{\mathsf{G}} = \{f \in R | g \cdot f = f\}$$

If the group G is finite we can turn every function into an invariant one:

Definition 3.1. The *Reynolds operator* is the projection map

$$\varrho : R \to S : f(x) \mapsto \frac{1}{\#\mathsf{G}} \sum_{g \in \mathsf{G}} g \cdot f.$$

This map is not a morphism of rings but it is a morphism of $\mathbb{C}[V]^{\mathsf{G}}$-modules (check this).

Theorem 3.2. *If G is a finite group and V a finite dimensional representation then the ring of invariants $S = \mathbb{C}[V]^{\mathsf{G}}$ is finitely generated.*

Proof. To prove that S is finitely generated, we first prove that this ring is Noetherian. Suppose that

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \mathfrak{a}_3 \subset \cdots$$

is an ascending chain of ideals in S. Multiplying with $R = \mathbb{C}[V]$ we obtain a chain of ideals in R:

$$\mathfrak{a}_1 R \subset \mathfrak{a}_2 R \subset \mathfrak{a}_3 R \subset \cdots .$$

This chain is stationary because R is a polynomial ring and hence Noetherian. Finally, $\varrho(\mathfrak{a}_i R) = \mathfrak{a}_i \varrho(R) = \mathfrak{a}_i S = \mathfrak{a}_i$ so the original chain must also be stationary.

Now let S_+ denote the ideal of S generated by all homogeneous elements of nonzero degree. Because S is Noetherian, S_+ is generated by a finite number of homogeneous elements: $S_+ = f_1 S + \cdots + f_r S$. We will show that these f_i also generate S as a ring.

Now $S = \mathbb{C} + S_+$ so $S_+ = \mathbb{C} f_1 + \cdots + \mathbb{C} f_r + S_+^2$, $S_+^2 = \sum_{i,j} \mathbb{C} f_i f_j + S_+^3$ and by induction

$$S_+^t = \sum_{i_1 \dots i_t} \mathbb{C} f_{i_1} \cdots f_{i_t} + S_+^{t+1}.$$

So $\mathbb{C}[f_1, \dots, f_r]$ is a graded subalgebra of S and $S = \mathbb{C} + S_+ = \mathbb{C}[f_1, \dots, f_r] + S_+^t$ for every t. If we look at the degree d-part of this equation we see that

$$S_d = \mathbb{C}[f_1, \dots, f_r]_d + (S_+^t)_d.$$

Because S_+^t only contains elements of degree at least t, $(S_+^t)_d = 0$ if $t > d$. As the equation holds for every t we can conclude that

$$S_d = \mathbb{C}[f_1, \dots, f_r]_d \text{ and thus } S = \mathbb{C}[f_1, \dots, f_r]$$

□

Now because S is finitely generated and does not have nilpotent elements, it corresponds to a variety $\mathbb{V}(S)$ and the embedding $S \subset R$ gives a map

$$\pi : \mathbb{V}(R) \to \mathbb{V}(S) : \mathfrak{m} \mapsto \pi(\mathfrak{m}) = \mathfrak{m} \cap S$$

which is a surjection because if $\mathfrak{s}$ is a maximal ideal in S then $\mathfrak{s}R$ is not equal to R. Indeed if $\mathfrak{s}R = R$ then $\mathfrak{s}$ contains an element $s$that is invertible in R. Just like s this inverse s^{-1} must also be invariant and hence it sits in S, which would imply that $\mathfrak{s} = S$. Therefore $\mathfrak{s}R$ will be contained in at least one maximal ideal $\mathfrak{m} \lhd R$, so $\pi(\mathfrak{m}) = \mathfrak{s}$. Furthermore if $\mathfrak{m} \lhd R$ then $\mathfrak{m} \cap S = g \cdot \mathfrak{m} \cap S$ so points of $\mathbb{V}(R)$ in the same orbit are mapped to the same point in $\mathbb{V}(S)$.

Vice versa if we have two disjoint orbits $\mathsf{G}p$ and $\mathsf{G}q$, we have a finite number of points in V and using interpolation we can construct a polynomial f which assigns to all points in $\mathsf{G}p$ the value 0 and to all points in $\mathsf{G}q$ the value 1. The polynomial $\varrho(f)$ will sit in S and will map the orbits $\mathsf{G}p$ and $\mathsf{G}q$ to different complex numbers so these two orbits must correspond to different points in $\mathbb{V}(S)$.

In this view it makes sense to define

Definition 3.3. If V is a finite dimensional representation of a finite group G then the quotient variety $V/\!\!/\mathsf{G}$ is the spectrum of the ring of invariants.

$$V/\!\!/\mathsf{G} := \mathbb{V}(\mathbb{C}[V]^{\mathsf{G}}).$$

The points in $V/\!\!/\mathsf{G}$ are in one-to-one correspondence with the orbits in V.

For more general (infinite) groups we can still define the ring $\mathbb{C}[V]^{\mathsf{G}}$ but sometimes this ring is not finitely generated. There is a special type of groups called *reductive groups*, for which $\mathbb{C}[V]^{\mathsf{G}}$ is finitely generated. Many well known groups are reductive $\mathsf{GL}_n(\mathbb{C}), \mathsf{SO}_n(\mathbb{C}), \dots$ but $(\mathbb{Z}, +)$ is not reductive.

For a reductive group it makes sense to define $V/\!\!/\mathsf{G} := \mathbb{V}(\mathbb{C}[V]^{\mathsf{G}})$ and there is a natural map $\pi : V \to V/\!\!/\mathsf{G}$ coming from the embedding $\mathbb{C}[V]^{\mathsf{G}} \subset \mathbb{C}[V]$. This map has a universal property

Lemma 3.4. *If $\phi : V \to \mathbb{X}$ is a morphism in $\mathtt{Aff-var}$ such that $\forall g \in \mathsf{G} : \phi(g \cdot x) = \phi(x)$ then there is a morphism $\tilde{\phi} : V/\!\!/\mathsf{G} \to \mathbb{X}$ such that $\phi = \tilde{\phi} \circ \pi$. (So in words every morphism in the category of affine varieties that kills the action factors through this map.)*

Proof. If the map $\phi_* := \mathbb{C}[\mathbb{X}] \to \mathbb{C}[V]$ kills the action then the image of ϕ_* is in $\mathbb{C}[V]^{\mathsf{G}}$ so we see ϕ_* as the composition of a map that has target $\mathbb{C}[V]^{\mathsf{G}}$ with the embedding $\mathbb{C}[V]^{\mathsf{G}} \subset \mathbb{C}[V]$. $\square$

For this reason the map $\pi : V \to V /\!\!/ \mathsf{G}$ is called the *categorical quotient*: it is the best map in the category of affine varieties that factors out the action. Note however that unlike in the finite group case, the points are not in 1-1 correspondence with the orbits. This is because it is possible that 2 orbits can have intersecting closures. If that is the case these two orbits must be mapped to the same point under π because π is continuous. Therefore $V /\!\!/ \mathsf{G}$ only classifies the closed orbits.

A simple example of this phenomenon is the $\mathbb{C}^*$-action on $V = \mathbb{C}^n$ by multiplication. This gives a $\mathbb{C}^*$-action on $\mathbb{C}[V]$ and the only invariant functions are the constant functions. The categorical quotient, $V /\!\!/ \mathbb{C}^* = \mathbb{V}(\mathbb{C})$, is just one point because there is only one closed orbit: the zero point. All other orbits have the zero point in their closure. So sometimes the categorical quotient does not provide enough information and one needs to construct other quotients.

3.2 Finite subgroups of $\mathsf{SL}_2(\mathbb{C})$

In this section we will classify some finite groups that will be important in the study of singularity theory.

Theorem 3.5. *Every finite subgroup of* $\mathsf{SL}_2(\mathbb{C})$ *can be conjugated to one of the following groups:*[1]

- A_n *a cyclic group* $\mathsf{C}_{\mathsf{n+1}}$ *with order* $n+1$ *generated by* $\begin{bmatrix} e^{\frac{2\pi i}{n+1}} & 0 \\ 0 & e^{\frac{-2\pi i}{n+1}} \end{bmatrix}$
- D_n *The binary dihedral group* $\mathsf{BD}_{\mathsf{n-2}}$ *with order* $4(n-2)$ *generated by* $\begin{bmatrix} e^{\frac{\pi i}{n-2}} & 0 \\ 0 & e^{\frac{-\pi i}{n-2}} \end{bmatrix}$ *and* $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
- E_6 *The binary tetrahedral group* BT *with order* 24.
- E_7 *The binary octahedral* BO *group with order* 48.
- E_8 *The binary icosahedral* BI *group with order* 120.

Proof. First note that if G is a finite subgroup of $\mathsf{SL}_2(\mathbb{C})$ we can conjugate it to a subgroup of SU_2. To prove this we can define a hermitian form on $\mathbb{C}^2$ as follows:

$$\langle v, w\rangle := \frac{1}{\#G}\sum_{g\in\mathsf{G}}(g\cdot v)(g\cdot w)^\dagger$$

[1] The numbering of the cases will become apparent later in the notes

The action of G keeps this form invariant: $\langle hv, hw\rangle = \sum_{g\in\mathsf{G}}(gh\cdot v)(gh\cdot w)^\dagger = \langle v,w\rangle$, so if we choose an orthonormal basis for this form G will act as unitary matrices according to this basis.

The group SU_2 can be mapped onto $\mathsf{SO}_3(\mathbb{R})$. Embed $\mathbb{R}^3$ in $\mathtt{mat}_2(\mathbb{C})$ as the subspace of traceless antihermitian matrices $\mathbb{H}$

$$\mathbb{R}\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]+\mathbb{R}\left[\begin{smallmatrix}0&i\\i&0\end{smallmatrix}\right]+\mathbb{R}\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right].$$

On this subspace we can put a scalar product $\langle A,B\rangle := Tr(AB^\dagger)$. SU_2 acts on this subspace by conjugation and the conjugation respects the scalar product: $\langle U\cdot A, U\cdot B\rangle = Tr(UAU^{-1}(UBU^{-1})^\dagger) = Tr(UAU^{-1}(UBU^{-1})^\dagger) = Tr(UAB^\dagger U^{-1}) = Tr(AB^\dagger)$. Therefore the action of SU_2 on $\mathbb{H}$ factors through the orthogonal group of $\langle,\rangle$. As SU_2 is connected the image of SU_2 will be contained in SO_3.

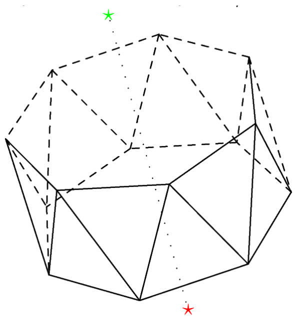
One can check that the kernel of this map is $\{1,-1\}\subset\mathsf{SU}_2$ and as the real dimension of SU_2 and $\mathsf{SO}_3(\mathbb{R})$ are both 3 the map will be surjective. SU_2 is called the double cover of SO_3.

Now we will show that any finite subgroup of SO_3 is either a cyclic group $\mathsf{C_n}$, a dihedral group $\mathsf{D_n}$ or one of the symmetry groups of a Platonic solid.

Let G be a finite subgroup of SO_3 with order n. The elements of $\mathsf{G}\setminus\{1\}$ are rotations so we can associate to each element its poles i.e. the intersection points of the rotation axis with the unit sphere. Let P be the set of poles of elements of G. Every pole is mapped to a pole under the action of G. So we can partition P into orbits of G. To every pole p we can associate m_p, the number of rotations with this pole. If we assume that every point is a pole of the trivial rotation, then m_p is also the order of the subgroup of G that fixes p. Note that poles in the same orbit have the same m_p.

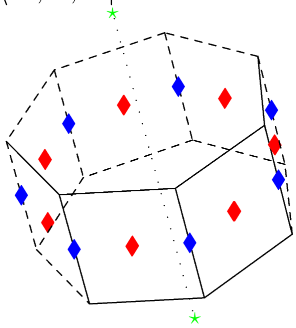
The $n-1$ non-trivial rotations in G consist of m_p-1 rotations for each pair of poles. That is $\frac{1}{2}(m_p-1)\frac{n}{m_p}$ for each orbit because an orbit has n/m_p poles and every rotation has 2 poles. Hence $n-1=\frac{1}{2}n(\sum\frac{(m_p-1)}{m_p})$ where the summation is over the orbits in P. Since $m_p\geq 2$ we have $(m_p-1)/m_p > 1/2$ and so we can only have 2 of 3 orbits if G is non-trivial.

1. *The case of two orbits.* Suppose these have n/m_1 and n/m_2 elements. Then $2/n = 1/m_1+1/m_2$ implies $n/m_1 = n/m_2 = 1$ and we have two orbits with one pole in each. This is the case when G is a cyclic group $\mathsf{C_n}$ generated by rotation by $2\pi/n$.

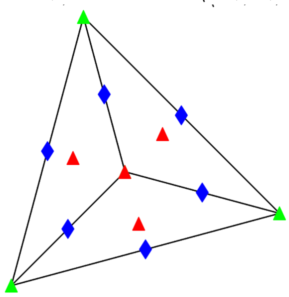


2. *The case of three orbits.* Then $1 + 2/n = 1/m_1 + 1/m_2 + 1/m_3$ so one of the $m_i = 2$. Take $m_3 = 2$ so $1/m_1 + 1/m_2 = 1/2 + 2/n$. There are only a few possibilities:

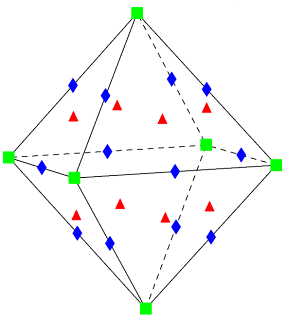
   - $m_1 = 2, m_2 = m, n = 2m$ (This is the dihedral case $\mathsf{G} = \mathsf{D}_{2n} = \langle X, Y, Z | X^2 = Y^m = Z^2 = XYZ = 1\rangle$)

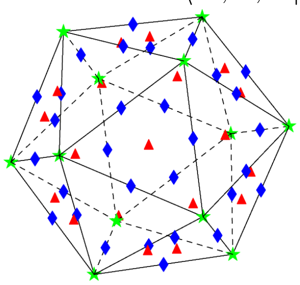
   - $m_1 = 3, m_2 = 3, n = 12$ (This is the symmetry group of the tetrahedron, $\mathsf{G} = \mathsf{T} = \langle X, Y, Z | X^3 = Y^3 = Z^2 = XYZ = 1\rangle$)

   - $m_1 = 3, m_2 = 4, n = 24$ (This is the symmetry group of the cube, $\mathsf{G} = \mathsf{O} = \langle X, Y, Z | X^3 = Y^4 = Z^2 = XYZ = 1\rangle$)



- $m_1 = 3, m_2 = 5, n = 60$ (This is the symmetry group of the dodecahedron $\mathsf{G} = \mathsf{I} = \langle X, Y, Z | X^3 = Y^3 = Z^2 = XYZ = 1\rangle$)



Now let $\tilde{\mathsf{G}}$ be a subgroup of SU_2. If $\tilde{\mathsf{G}}$ has an even number of elements then it contains -1, because this is the only element in SU_2 of order 2. This means that $\tilde{\mathsf{G}}$ is the inverse image of a finite subgroup of SO_3, these are called the binary dihedral, tetrahedral, etc. groups, note that binary cyclic is again cyclic. These groups can be expressed in generators and relations by introducing a new generator T that commutes with all others and $T^2 = 1$, the relations of the original group are then put equal to T instead of one.

- the binary dihedral case $\mathsf{G} = \mathsf{BD}_n = \langle X, Y, Z | X^2 = Y^m = Z^2 = XYZ = T, T^2 = 1\rangle$
- the binary tetrahedral case $\mathsf{G} = \mathsf{BT} = \langle X, Y, Z | X^3 = Y^3 = Z^2 = XYZ = T, T^2 = 1\rangle$
- the binary octahedral case $\mathsf{G} = \mathsf{BO} = \langle X, Y, Z | X^3 = Y^4 = Z^2 = XYZ = T, T^2 = 1\rangle$
- the binary dodecahedral case $\mathsf{G} = \mathsf{BI} = \langle X, Y, Z | X^3 = Y^3 = Z^2 = XYZ = T, T^2 = 1\rangle$

If $\tilde{G}$ has an odd number of elements then it is isomorphic to its image, which must be cyclic because all other subgroups of SO_3 have even order. $\square$

3.3 Kleinian singularities as quotient singularities

In this section we will determine generators and relations for the rings of invariants $\mathbb{C}[V]^{\mathsf{G}}$.

Theorem 3.6. *Let G be a finite subgroup of $\mathsf{SL}_2(\mathbb{C})$ and $V = \mathbb{C}^2$ its standard representation then the ring of invariants $\mathbb{C}[V]^{\mathsf{G}}$ is isomorphic to*

$$\mathbb{C}[X, Y, Z]/(f)$$

where f is

A_n $XY - Z^{n+1}$ *if* $\mathsf{G} \cong \mathsf{C_{n+1}}$,

D_n $X^{n-1} - XY^2 + Z^2$ *if* $\mathsf{G} \cong \mathsf{BD_{n-2}}$,

E_6 $X^4 + Y^3 + Z^2$ *if* $\mathsf{G} \cong \mathsf{BT}$,

E_7 $X^3Y + Y^3 + Z^2$ *if* $\mathsf{G} \cong \mathsf{BO}$,

E_8 $X^5 + Y^3 + Z^2$ *if* $\mathsf{G} \cong \mathsf{BI}$.

In other words the quotient varieties of finite subgroups of $\mathsf{SL}_2(\mathbb{C})$ are precisely the Kleinian singularities.

Proof. In order to prove this we use the Reynolds operator,

$$\varrho(f) = \frac{1}{|G|} \sum_{g \in \mathsf{G}} f^g.$$

This map is a projection $\varrho^2 = \varrho$ and it is the identity operation on $\mathbb{C}[V]^{\mathsf{G}}$. So to get a basis for the ring of invariants we can look at the set of images of all the monomials in $\mathbb{C}[V]$.

$$\varrho X^i Y^j.$$

We will only work out the case for A_n, the computations for the other groups are best done using a computer algebra package like GAP. If g is the generator of the cyclic group then $g \cdot X = \zeta X, g \cdot Y = \zeta^{-1} Y$ with $\zeta = e^{2\pi i/n+1}$. Therefore

$$\begin{aligned}
\varrho X^i Y^j &= \frac{1}{n+1} \sum_{k=0}^{n} g^k \cdot X^i Y^j \\
&= \frac{1}{n+1} \sum_{k=0}^{n} \zeta^{k(i-j)} X^i Y^j \\
&= \begin{cases} 0 & i \neq j \mod n+1 \\ X^i Y^j & i = j \mod n+1 \end{cases}.
\end{aligned}$$

From this one can deduce that all invariants are generated by X^{n+1}, Y^{n+1} and XY. We get a surjective map

$$\pi : \mathbb{C}[X,Y,Z] \to \mathbb{C}[X,Y]^{\mathsf{G}} : f(X,Y,Z) \mapsto f(X^{n+1}, Y^{n+1}, XY).$$

The kernel of this map clearly contains $XY - Z^{n+1}$. The dimension of the quotient space must be two because the map $\mathbb{C}^2 \to V /\!\!/ \mathsf{G}$ has finite fibers. If the ideal were generated by more than one generator, the corresponding variety would not be two-dimensional. □

3.4 The skew group ring

Now that we have identified the Kleinian singularities as quotient singularities, we can use the group action to embed the ring of invariants into a bigger non-commutative algebra. The action of G on a vector space V gives rise to an action on the polynomial ring $\mathbb{C}[V]$. Similar to the group algebra we can now construct the skew group ring.

Definition 3.7. The *skew group ring* or *smash product* consists of all $\mathbb{C}[V]$-linear combinations of group elements

$$\mathbb{C}[V] \star G = \{\sum_{g \in \mathsf{G}} f_g g | f_g \in \mathbb{C}[V]\}$$

We can define a product on this vector space

$$f_g g \cdot f_h h = (f_g(g \cdot f_h))gh,$$

and linearly extend it to the whole vector space. In this expression the $(g \cdot f_h)$ denotes the action of g on $f_h \in \mathbb{C}[V]$.

The center of this algebra can be easily determined: if $z = \sum_g f_g g \in Z(\mathbb{C}[V] \star G)$ then

$$\forall f \in \mathbb{C}[V] : [z, f] = \sum_g f_g(f - g \cdot f)g = 0 \text{ and } \forall h \in G : [z, h] = \sum_g (h \cdot f_g - f_g)gh$$

The first equation implies that $f_g = 0$ if $g \neq 1$ and the second implies that f_1 must be a G-invariant function so we can conclude that

Lemma 3.8. $Z(\mathbb{C}[V] \star \mathsf{G}) \cong \mathbb{C}[V]^{\mathsf{G}}$.

The algebra $A = \mathbb{C}[V] \star G$ has a natural grading that assigns degree 1 to X and Y and degree 0 to all group elements. The degree 0 elements form a subalgebra A_0. This algebra is isomorphic to the group algebra $\mathbb{C}G$.

Theorem 3.9 (Artin-Wedderburn). *$A_0 = \mathbb{C}\mathsf{G}$ is a direct sum of $n_i \times n_i$ matrix algebras, where the n_i are the dimensions of the simple representations of G.*

$$\mathbb{C}\mathsf{G} \cong \bigoplus_{i=1}^{k} \mathtt{mat}_{n_i \times n_i}(\mathbb{C})$$

Proof. The proof can be found in every book on representations of finite groups. □

This isomorphism provides a basis of the form E^i_{rs} which are the elementary matrices in the i^{th} block with a 1 on the r, s-entry and zero everywhere else.

Now let $e = \sum_i E^i_{11}$ denote the element that corresponds to the matrix having a 1 in the upper left corner for each representation and zeros everwhere else. This element is the sum of k idempotent elements E^i_{11}, which we denote by $e_1, \dots, e_k$. Each of these corresponds to a unique simple representation of G. These representations can be seen as $W_i \cong \mathbb{C}Ge_i$ The e_i also have the property that for a $\mathbb{C}\mathsf{G}$-representation W the dimension of the subspace e_iW is the same as the multiplicity of W_i inside W. Another important property is that the ideal generated by e is the full group algebra, $\mathbb{C}\mathsf{G}e\mathbb{C}G = \mathbb{C}\mathsf{G}$, this is because matrix algebras have no proper ideals and e has a nonzero value in every matrix component of $\mathbb{C}\mathsf{G}$. Given the algebra $A = \mathbb{C}[V] \star \mathsf{G}$, we look at the subspace $\Pi := eAe$. This space is again an algebra but its unit element is now e instead of 1. This algebra is smaller than A but it still keeps the same information.

Theorem 3.10. *A and eAe are Morita equivalent.*

Proof. $AeA = A$ because $\mathbb{C}\mathsf{G}e\mathbb{C}G = \mathbb{C}\mathsf{G}$. □

So instead of looking at the representations of A we can consider representations of Π. Π has the advantage that it is an algebra over $e\mathbb{C}\mathsf{G}e = \mathbb{C}^{\oplus k}$ so it is the quotient of a path algebra of a quiver.

3.5 McKay Quivers

To apply the theory to the situation we're interested in, we need an explicit description of Π in terms of its quiver and its relations.

Because $\Pi = eAe$ and e is a direct sum of elementary matrices, one for each simple representation, The vertices of the quiver for Π will correspond to the simple representations of G.

Because A is generated by A_1 as an algebra over $A_0 = \mathbb{C}G$ and $AeA = A$, the algebra Π will be generated by eA_1e. The arrows will correspond to generators of eA_1e, these sit inside the degree 1 part: $A_1 = V \otimes \mathbb{C}G = (\mathbb{C}X + \mathbb{C}Y)\mathbb{C}G$.

For every couple idempotents e_i, e_j we can choose a basis for the subspace $e_i(\mathbb{C}X + \mathbb{C}Y)\mathbb{C}Ge_j \subset \mathbb{C}[V] \star G$. The union of all these bases forms a basis $\{a_1, \ldots, a_l\}$ for the space eA_1e. We can now construct a quiver Q_G with vertices the set $\{e_1, \ldots, e_k\}$ and as arrows $\{a_1, \ldots, a_l\}$. If the arrow a_ℓ sits in $e_i(\mathbb{C}X + \mathbb{C}Y)\mathbb{C}Ge_j$ then we let it run from e_j to e_i: $h(a_\ell) = e_i, t(a_\ell) = e_j$.

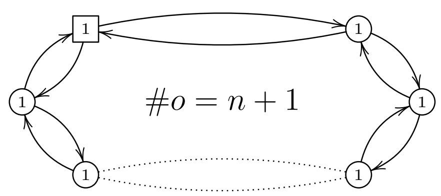
Lemma 3.11. *The dimension of $e_i(\mathbb{C}X + \mathbb{C}Y)\mathbb{C}Ge_j$ is the multiplicity of the simple representation W_i inside $(\mathbb{C}X + \mathbb{C}Y) \otimes W_j$.*

Proof. From the representation theory of finite groups, we know that $\mathbb{C}G$ is isomorphic to $\bigoplus_i \mathsf{End}_{\mathbb{C}}(W_i)$. The element e_i is an idempotent in $\mathsf{End}_{\mathbb{C}}(S_i)$ that projects the representation S_i onto a one-dimensional subspace and acts as zero on the other simple representations. Therefore if W is any representation, the dimension of e_iW will be the multiplicity of S_i inside W. On the other hand it is easy to check that $\mathbb{C}Ge_j$ is isomorphic to W_j as a G-representation. $\square$

Definition 3.12. Let V be a vector space and G be a finite subgroup of $GL(V)$. The *McKay quiver* of (G, V) is the quiver of which the vertices correspond to the simple representations of G and the number of arrows from W_j to W_i is the multiplicity of W_i inside $V \otimes W_j$.

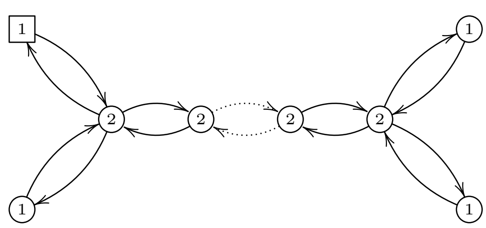
Theorem 3.13. *The McKay quiver of the $G \subset \mathsf{SL}_2(\mathbb{C})$ looks as follows. In the vertices we have put the dimensions of the corresponding representations of* G. *We put a square around the vertex corresponding to the trivial representation.*

$\tilde{A}_n$



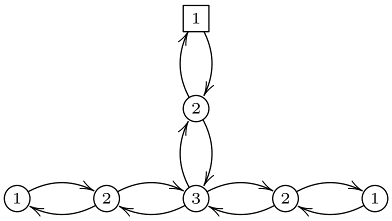
if $\mathsf{G} \cong \mathsf{C}_{n+1}$,

$\tilde{D}_n$



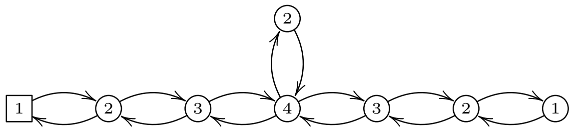
if $\mathsf{G} \cong \mathsf{BD}_{n-2}$,

$\tilde{E}_6$



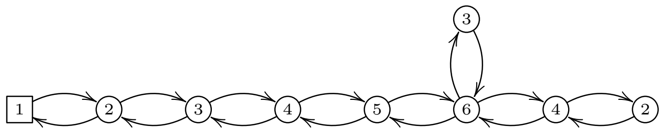
if $\mathsf{G} \cong \mathsf{BT}$,

$\tilde{E}_7$



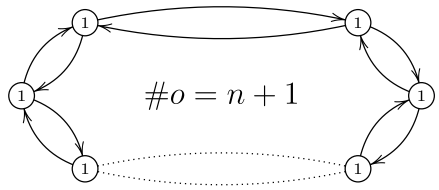
if $\mathsf{G} \cong \mathsf{BO}$,

$\tilde{E}_8$



if $\mathsf{G} \cong \mathsf{BI}$.

Proof. We only do the A_n case. $\mathsf{C_{n+1}}$ is a cyclic group generated by g. It has $n+1$ simple one-dimensional representations corresponding to the $n+1^{th}$ roots of 1. $\chi_{S_k}(g) = e^{2k\pi i/n+1}$. $V = W_0 \oplus W_n$ and $W_i \otimes W_j = W_{i+j}$ where the sum is modulo $n+1$. The representation V is isomorphic to $W_1 \oplus W_{-1}$ so every vertex is connected to two others and the McKay quiver looks like:



□

If we forget about the orientations of the arrows in Q and treat a pair of arrows that run in opposite directions as an edge, we get a graph. These graphs are known as the (simply laced) *extended Dynkin diagrams*. They look like the ordinary Dynkin diagrams but have one extra node. This node corresponds to the trivial representation. The extended Dynkin diagrams are denoted by $\tilde{A}_n, \tilde{D}_n, \tilde{E}_{6,7,8}$. Note that the subscript is one less than the number of nodes in the extended Dynkin diagram.

3.6 Preprojective algebras

Given a quiver Q we define the *double quiver* $\bar{Q}$ as the quiver with the same vertices and arrows as Q but for each arrow $a \in Q_1$ an extra arrow a^* in the opposite direction ($h(a^*) = t(a)$ and $t(a^*) = h(a)$).

In the path algebra of a double quiver we can define a special element in Π:

$$\omega := \sum aa^* - a^*a$$

where the sum runs over the arrows in Q.

Definition 3.14. The *preprojective algebra* of Q is the quotient of the path algebra of the double quiver $\bar{Q}$ by $\langle\omega\rangle$:

$$\Pi(Q) = \frac{\mathbb{C}\bar{Q}}{\langle\omega\rangle}.$$

If $\lambda \in \mathbb{C}^{Q_0}$, the *deformed preprojective algebra* is defined as

$$\Pi^\lambda(Q) = \frac{\mathbb{C}\bar{Q}}{\langle\omega - \sum_{v\in Q_0}\lambda_v v\rangle}.$$

The representation theory of this algebra highly depends on the quiver.

Theorem 3.15 (Crawley-Boevey)**.** *The algebra $\Pi^\lambda(Q)$ is*

- *finite dimensional if Q is a quiver whose underlying graph is a Dynkin diagram,*
- *infinite dimensional but Noetherian if Q is a quiver whose underlying graph is an Extended Dynkin diagram.*
- *non-Noetherian otherwise.*

Proof. See [**?**]. □

Theorem 3.16. *If G is a finite subgroup of $\mathsf{SL}_2(\mathbb{C})$ and $V = \mathbb{C}^2$ then*

$$e\mathbb{C}[V] \star \mathsf{G}e \cong \Pi(Q)$$

where Q is a quiver whose underlying graph is the corresponding extended Dynkin diagram.

Proof. We only do the proof for A_n. In that case Π is isomorphic to $\mathbb{C}[X,Y]\star G$ because all representations are one-dimensional. The McKay quiver is a cycle: The unstarred arrows are the clockwise arrows and correspond to e_iXe_{i+1} while the starred arrows are the anticlockwise and correspond to $e_{i+1}Ye_i$. The element $(XY-YX)$ in $\mathbb{C}\langle X,Y\rangle\star G$ is precisely ω and evaluates to zero in $\mathbb{C}[X,Y]\star G$, so so there is a surjective map $\mathbb{C}Q/\langle\omega\rangle\to\mathbb{C}[X,Y]\star G$. To show that it is bijective one can compare the dimensions of the degree k-components of $\mathbb{C}Q/\langle\omega\rangle$ and Π.

The degree k component of Π is $\mathbb{C}[X,Y]_k\star G$ which has dimension $(k+1)|G|$. The degree k component of $\mathbb{C}Q$ consist of all paths of length k in each vertex you can leave either via a starred or an unstarred arrow. The relation $e_i\omega = e_iXe_{i+1}e_{i+1}Ye_i - e_iYe_{i-1}e_{i-1}Xe_i$ allows us to swap stars, so we can put all stars at the beginning. For each vertex there are at most $k+1$ paths of length k (depending on the number of starred arrows). As there are $|G|$ vertices the dimension of the dimension of $\mathbb{C}Q/\langle\omega\rangle_n$ is at most $(k+1)|G|$. Therefore $\mathbb{C}Q/\langle\omega\rangle\to\Pi$ must be a bijection in each degree. $\square$

4

Resolutions of Singularities

4.1 Graded Rings

A ring R is graded if it can be written as a direct sum

$$R = R_0 \oplus R_1 \oplus \dots$$

such that $R_i R_j \subset R_{i+j}$. Recall that by an infinite direct sum $R = \bigoplus_{i=0}^{\infty} R_i$ we mean that every element in R can be written in a unique way as as sum $r = \sum_{i=0}^{\infty} r_i$ with $r_i \in R_i$ and only a finite number of r_i are nonzero. An element r is called homogeneous of degree i if it sits in R_i.

Rings can often be graded in different ways. If $R = \mathbb{C}[X_1, \dots, X_n]$ we can assign to each X_i a degree n_i and then we put $R_i = \langle X_1^{e_1} \dots X_n^{e_n} | e_1 n_1 + \dots + e_n n_n = i \rangle$, so R_i is spanned by all monomials of degree i.

An ideal $\mathfrak{m}$ of a graded ring R is called graded if $\mathfrak{m} = \bigoplus_{i=0}^{\infty} \mathfrak{m}_i$ with $\mathfrak{m}_i = R_i \cap \mathfrak{m}$. Note that if $\mathfrak{m}$ is graded then $R/\mathfrak{m}$ is also graded with $(R/\mathfrak{m})_i = R_i/\mathfrak{m}_i$. By definition a graded ideal is generated by homogeneous elements, and the converse is also true: if the generators $r_1, \dots, r_k$ are homogeneous with degrees $e_1, \dots e_k$ then $\mathfrak{m}_i = R_{i-e_1} r_1 + \dots R_{i-e_k} r_k$

This makes it easy to construct graded rings: we can write them as quotients

$$R = \mathbb{C}[X_1, \dots, X_n]/(r_1, \dots, r_k)$$

where the r_i are homogeneous polynomials for the grading induced by the degrees we assign to the X_i.

Now suppose that all variables of the graded ring R have degree 1. The affine variety of this ring, $\mathbb{V}(R)$, wil consist of lines through the origin because if $r_i(X_1, \dots, X_n)$ is homogeneous then $r_i(\lambda X_1, \dots, \lambda X_n) = \lambda^{\deg r_i} r_i(X_1, \dots, X_n)$. This means that $\mathbb{V}(R)$ is a cone with the origin as its top.

The converse also holds if $\mathbb{X} \subset \mathbb{C}^n$ is a cone through the origin, we can define a group action of $\mathbb{C}^*$ on $\mathbb{X}$ by scaling:

$$\mathbb{C}^* \times \mathbb{X} \to \mathbb{X} : (\lambda, x) \mapsto \lambda x$$

This action also gives an action on $\mathbb{C}[\mathbb{X}]$:

$$\mathbb{C}^* \times \mathbb{C}[\mathbb{X}] \to \mathbb{C}[\mathbb{X}] : (\lambda, f(x)) \mapsto f(\lambda x)$$

To turn $\mathbb{C}[\mathbb{X}]$ into a graded ring we define

$$\mathbb{C}[\mathbb{X}]_i := \{f \in \mathbb{C}[\mathbb{X}] | f(\lambda x) = \lambda^i f(x)\}.$$

4.2 Non-affine varieties

Up until now we have only seen affine varieties, however one can construct far bigger class of varieties by gluing affine varieties together. We will not attempt to formulate a complete abstract definition of an algebraic variety but the main idea is that if $\mathbb{V}$ is a variety we can see it as the union of some affine varieties that are identified on affine Zariski-open subsets. More precisely we have affine varieties $\mathbb{V}_i$ and affine embeddings $\iota_{ij} : \mathbb{V}_{ij} \subset \mathbb{V}_i$ such that $\mathbb{V}_{ij} = \mathbb{V}_{ji}$ and the images $\iota_{ij}\mathbb{V}_{ij}$ are zariski open subsets of $\mathbb{V}_i$. The full variety $\mathbb{V}$ is then defined as

$$\mathbb{V} = \bigcup_i \tilde{\mathbb{V}}_i / \sim,$$

where $x_i \sim x_j$ for $x_i \in \mathbb{V}_i$ if $\iota_{ij} x_i = \iota_{ji} x_j$. A map between two such varieties is a morphism of varieties if the maps between the restrictions to the affine open subsets are morphisms of affine varieties.

The simplest example is $\mathbb{P}_n$, which is the set of lines through the origin in $\mathbb{C}^{n+1}$

$$\{\mathbb{C}(x_0, \dots, x_n) | \exists i : x_i \neq 0\}.$$

Each line is considered as a point in projective space which we denote by $(x_1 : \dots : x_{n+1})$. The coordinates of the point are defined up to a scalar. We can cover $\mathbb{P}_n$ by $n+1$ affine spaces

$$\mathbb{A}_n^{(i)} = \{(x_0 : \dots : x_n) | x_i \neq 0\} \cong \mathbb{C}^n = \{\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_m}{x_i}\}.$$

Just like for affine varieties we can make subvarieties by looking at the zeros of polynomials, but to be well defined these polynomials must be homogeneous i.e. $f(\lambda X_1, \dots, \lambda X_{n+1}) = \lambda^k f(X_1, \dots, X_n)$. This is because we want that the full line through the origin in $\mathbb{C}^{n+1}$ is zero for f or not. This can be formalized by the proj construction.

If $A = \mathbb{C}[Y_1, \dots, Y_n]/(r_1, \dots, r_k)$ is a positively graded ring generated by $Y_0, \dots, Y_n$ with degree 1 then the variety $\mathbb{V}(A) \subset \mathbb{C}^{n+1}$ is a union of lines through the origin and the points at infinity through these lines form a subset of $\mathbb{P}_n$, which we denote by

$$\mathbb{P}A := \{(y_0 : \cdots : y_n) \in \mathbb{P}_n | r_i(y_0, \dots, y_n) = 0\}.$$

Such varieties are called projective varieties.

Using the $\mathbb{P}$-construction we can also make a mixture between affine and projective varieties nl. the quasi-projective varieties. If $A = A_0 \oplus A_1 \oplus \dots$ is a positively graded ring generated by $X_1, \dots, X_n$ with degree 0 and $Y_0, \dots, Y_m$ with degree 1 and let $r_1, \dots r_p$ the homogeneous relations between the generators, then we can define a subset

$$\mathbb{P}A := \{(x_1, \dots, x_n, y_0 : \cdots : y_m) \in \mathbb{C}^n \times \mathbb{P}^m | r_i(x_1, \dots, x_n, y_0 : \cdots : y_m) = 0\}$$

This set is well defined because if r_i is of degree d then

$$r_i(x_1, \dots, x_n, \lambda y_0, \dots, \lambda y_m) = 0 \Leftrightarrow \lambda^d r_i(x_1, \dots, x_n, y_0 : \cdots : y_m) = 0.$$

Now we can cover $\mathbb{P}A$ by affine open subset $\mathbb{V}_i$ corresponding to the locus where y_i is nonzero:

$$\mathbb{V}_i = \{(x_1, \dots, x_n, \frac{y_0}{y_i}, \dots, \frac{y_{i-1}}{y_i}, \frac{y_{i+1}}{y_i}, \dots, \frac{y_m}{y_i}) | (x_1, \dots, x_n, y_0 : \cdots : y_m) \in \mathbb{P}A\}$$

Note that the coordinate ring of each affine part is the degree zero part obtained after inverting the i^{th} coordinate: $\mathbb{C}[\mathbb{V}_i] = A[y_i^{-1}]_0$ (Note that y_i^{-1} has degree -1). Furthermore $\mathbb{C}[\mathbb{V}_i \cap \mathbb{V}_j] = A[y_i^{-1}, y_j^{-1}]_0$ so $\mathbb{P}A$ consist of affine varieties $\mathbb{V}_i$ glued together by identifying affine Zariski-open subsets $\mathbb{V}_i \cap \mathbb{V}_j$ [1].

The variety $\mathbb{P}A$ can be mapped to the affine variety $\mathbb{V}(A_0)$ by the ordinary projection

$$(x_1, \dots, x_n, y_0, \dots, y_m) \mapsto (x_1, \dots, x_n)$$

and each of the fibers is a projective variety.

It is important to remark that unlike in the affine case, non-isomorphic graded rings can give rise to the same quasiprojective variety. The standard example

[1] If A is not only generated by elements in degree 0,1 it is still possible to define $\mathbb{P}A$ abstractly as the glueing of the $\mathbb{V}_i = \mathbb{V}(A[y_i^{-1}]_0)$, where $\deg y_i \neq 0$.

of this phenomenon is the conic and the projective line. Consider the rings $R = \mathbb{C}[X, Y]$ and $S = \mathbb{C}[X, Y, Z]/(XY - Z^2)$ where all variables have degree 1. $\mathbb{P}R$ can be seen as the projective line, while $\mathbb{P}S$ is the projective conic. These graded rings are not isomorphic because $\dim R_1 = 2$ and $\dim S_1 = 3$ (they are even not isomorphic as rings). There is however a bijective morphism

$$\mathbb{P}R \to \mathbb{P}S : (x : y) \mapsto (x^2 : y^2 : xy).$$

One of the advantages of working with projective varieties is that it is possible to construct varieties that behave like compact spaces. Note that over $\mathbb{C}$ every affine variety that is not a finite number of points, is noncompact for the standard complex topology. This is because the solutions to polynomials in $\mathbb{C}^n$ always form a nonbounded set. The space $\mathbb{P}^n$ however is compact for the standard complex topology, so all closed subsets of $\mathbb{P}_n$ are compact. In particular this means that $\mathbb{P}A$ is compact if A is generated by elements of degree 1. If A also has generators of degree 0, $\mathbb{P}A$ need not to be compact but the fibers of the standard map $\mathbb{P}A \to \mathbb{V}(A_0)$ are all compact. A map for which the preimage of a compact set is compact is called a proper map.

4.3 Resolutions of singularities

With all this in mind we can introduce the concept of a resolution of a singularity:

Definition 4.1. If $\mathbb{V}$ is a singular irreducible variety then a morphism of varieties $\pi : \mathbb{W} \to \mathbb{V}$ is a resolution of $\mathbb{V}$ if

- R1 π is a surjection,
- R2 π is a proper map,
- R3 π is almost everywhere one-to-one: there are open subsets $\mathbb{U}_\mathbb{V} \subset \mathbb{V}$ and $\mathbb{U}_\mathbb{W} \subset \mathbb{W}$ such that $\pi|_{\mathbb{U}_\mathbb{W}}$ is an isomorphism between $\mathbb{U}_\mathbb{W}$ and $\mathbb{U}_\mathbb{V}$,
- R4 $\mathbb{W}$ is an irreducible smooth variety.

The locus of $\mathbb{W}$ for which π is not one-to-one a called the *exceptional locus*. If $\mathbb{W}$ is not smooth but all the rest holds we call π a partial resolution.

The idea of a resolution is that we make $\mathbb{V}$ smooth by substituting a small part of it by something somewhat bigger. Everything in the original variety must be represented by something in the resolution, so π must be a surjection. We don't want to change too much so π must be almost everywhere one-to-one. Finally we want π to be proper because we want to change compact things by compact things. If we would allow noncompact fibers we could always make a resolution smaller by deleting a point in a noncompact fiber.

In dimension one we can construct a resolution resolution using a well known technique from number theory: *the integral closure*. If R is a domain and $K = \{f/g|f, g \in R, g \neq 0\}$ is its field of fractions then the *integral closure* of R is the set of elements in K that satisfy a monic polynomial with coefficients in R.

$$\tilde{R} = \{u \in K|\exists r_0, \dots, r_{k-1} \in R : u^k + r_{k-1}u^{k-1} + \dots + r_0 = 0\}$$

A ring that is equal to its own integral closure is also called *normal* and the integral closure is also called the *normalization*. Note that the integral closure has the same Krull dimension as the original ring because they have the same field of fractions.

For example if $R = \mathbb{C}[X, Y]/(X^2 - Y^3)$ then $Z := X/Y$ is integral over R because $Z^2 - Y = 0$. In this example one can show that the integral closure of R is $\mathbb{C}[Z]$. There is a standard embedding of $R \subset \tilde{R}$ because every element in R satisfies the monic polynomial $X - r$. This gives a map

$$\mathbb{V}(\tilde{R}) \to \mathbb{V}(R).$$

Theorem 4.2 (Zariski's main theorem)**.** *If $R = \mathbb{C}[\mathbb{X}]$ is an affine ring over $\mathbb{C}$ and $\mathfrak{m}$ a maximal ideal then the normalization of the completion is isomorphic to the completion of the normalization.*

$$\widetilde{\widehat{R}_{\mathfrak{m}}} \cong \widehat{\tilde{R}_{\mathfrak{m}}}$$

Proof. Zariski-Samuel II, chap. VIII, 13, pp. 313-320. □

Theorem 4.3 (Normality and Singularities)**.** *If $R = \mathbb{C}[\mathbb{X}]$ is a normal affine ring over $\mathbb{C}$ then the singular locus of $\mathbb{X}$ has dimension at most* $\dim \mathbb{X} - 2$.

Proof. Mumford, chap. III, sec. 8, p. 273; Shafarevich, p. 111. □

Theorem 4.4. *If $R = \mathbb{C}[\mathbb{X}]$ is the coordinate ring of a one-dimensional affine variety then the map*

$$\tilde{\mathbb{X}} = \mathbb{V}(\tilde{R}) \to \mathbb{V}(R) = \mathbb{X}.$$

is a resolution.

Proof. The map is surjective because if $\mathfrak{p}$ is a maximal prime in R then $\mathfrak{p}\tilde{R}$ is an ideal in $\tilde{R}$. This ideal is not $\tilde{R}$ otherwise an element $u \in \mathfrak{p}$ would have an inverse

$v \in \tilde{R}$. This inverse satisfies some minimal polynomial $f(X) = X^k + r_{k-1}X^{k-1} + \cdots + r_0 \in R[X]$ but then multiplying $f(v)$ with u^{k-1} we see that

$$u^{k-1}f(v) = v + r_{k-1} + r_{k-2}u + \cdots + r_0 u^{k-1} = 0,$$

and v must sit in R. Therefore $\mathfrak{p}\tilde{R}$ is contained in at least one maximal ideal of $\tilde{R}$.

The map is almost everywhere one to one because the two rings have the same field of fractions. Therefore the generators $\tilde{R}$ are of the form f_i/g_i with $f_i, g_i \in R$. So if we invert the g_i in R and $\tilde{R}$ we get isomorphic rings and there is a bijection between $\{gx \in \mathbb{V}(R)|g_i(x) \neq 0\}$ and $\{gx \in \mathbb{V}(\tilde{R})|g_i(x) \neq 0\}$.

The map is proper because every fiber contains only a finite number of points. To see this suppose that there are an infinite number of points on this fiber then there is a one-dimensional subvariety of $\mathbb{Y} \subset \tilde{\mathbb{X}}$ that is mapped to one point in $\mathbb{X}$. Because $\tilde{\mathbb{X}}$ is one-dimensional and irreducible the subvariety must be the whole of $\tilde{\mathbb{X}}$.

Finally, the singular locus of an integral ring with dimension n has dimension $n - 2$, so if $n = 1$ the singular locus must be empty. □

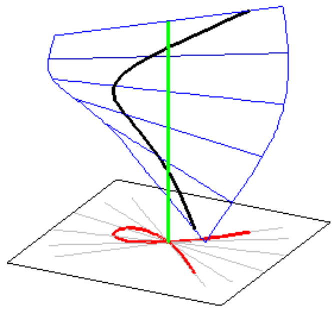
For example consider the ring $R = \mathbb{C}[X,Y]/(Y^2 - X^2(X-1))$. This ring corresponds to a curve with a single node at the origin. If we go to the integral closure we have to add an element $Z := Y/X$, which satisfies $Z^2 - (X-1)$. The resulting ring is

$$\mathbb{C}[X,Y,Z]/(Y^2 - X^2(X-1), ZX - Y, Z^2 - (X-1)) = \mathbb{C}[Z].$$

The resolution corresponds to the embedding

$$R \to \mathbb{C}[Z] : \begin{cases} X & \mapsto Z^2 + 1 \\ Y & \mapsto Z(Z^2 + 1) \end{cases}$$

The resolution is one-to-one everywhere except for $X = Y = 0$, where we can chose $Z = \pm 1$. These 2 points that map to the zero correspond to the two incoming branches in the singular point one with slope $Y/X = 1$ and one with slope $Y/X = -1$. The idea that you need to add directions of incoming tangent lines leads to the idea of blowing up a singularity.



4.4 Blow-ups

Definition 4.5. Suppose $\mathbb{V}$ is an affine variety and $\mathfrak{n} \lhd \mathbb{C}[\mathbb{V}]$ is an ideal corresponding to the closed subset $\mathbb{X}$. The blow-up of $\mathbb{X}$ is $\tilde{\mathbb{V}} = \mathbb{P}\tilde{R}$ with

$$\tilde{R} = \mathbb{C}[\mathbb{V}] \oplus \mathfrak{n}t \oplus \mathfrak{n}^2 t^2 \oplus \cdots \subset \mathbb{C}[\mathbb{V}][t].$$

The standard projection $\pi : \tilde{\mathbb{V}} \to \mathbb{V}$ is at least a partial resolution because it is a proper surjection and π is almoste everywhere one-to-one because if $p \in \mathbb{V} \setminus \mathbb{X}$ and $y_0 t, \ldots y_m t$ are the generators of $\mathfrak{n}t$, there must be at least one y_i that is not zero on p, so the preimage of p will only contain the point

$$(x_1(p), \ldots, x_n(p), y_0(p) : \cdots : y_m(p)).$$

The simplest example of a blow-up is the blow-up of the zero point in affine space $\mathbb{C}^n$. In that case $\mathfrak{m} = (X_1, \ldots, X_n) \lhd \mathbb{C}[X_1, \ldots, X_n]$ and

$$\tilde{R} = \mathbb{C}[X_1, \ldots, X_n, X_1 t, \ldots, X_n t] \subset \mathbb{C}[X_1, \ldots, X_n][t]$$

If we define $Y_i := X_i t$ we see that

$$\tilde{R} = \mathbb{C}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/(X_i Y_j - X_j Y_i)$$

so

$$\tilde{\mathbb{C}}^n := \{(x_1, \ldots, x_n, y_1 : \cdots : y_n) \in \mathbb{C}^n \times \mathbb{P}^{n-1} | x_i y_j = x_j y_i\}.$$

If $(x_1, \ldots, x_n) \neq 0$ then the y_i are fixed up to a scalar so the map $\pi : \tilde{\mathbb{C}} \to \mathbb{C}$ is one to one for these points. For the zero point the y_i can be whatever they want so the exceptional locus is $\{0\} \times \mathbb{P}^{n-1}$. The points in this fiber represent all possible lines through the zero point.

If $\mathbb{V}$ is a closed affine subvariety of $\mathbb{C}^n$ containing the zero then we can also blow up this point. If we denote the maximal ideal in $R = \mathbb{C}[\mathbb{V}] = \mathbb{C}[X_1, \dots, X_n]/\mathfrak{f}$ by $\mathfrak{n}$, the blow-up is the proj of the ring $\tilde{R} = \mathbb{C}[\mathbb{V}] \oplus \mathfrak{n}t \oplus \mathfrak{n}^2 t^2 \oplus \dots$.

Again the blow-up is everywhere one to one except for the zero fiber. To find the zero fiber we need to divide out the ideal in $\tilde{R}$ generated by $X_1, \dots, X_n$. The part in $\mathfrak{n}^i t^i$ that is generated by $X_1, \dots, X_n$ is precisely $\mathfrak{n}^{i+1} t^i$ so

$$\tilde{R}/(X_1, \dots, X_n) = \mathbb{C} \oplus \frac{\mathfrak{n}}{\mathfrak{n}^2} t \oplus \frac{\mathfrak{n}^2}{\mathfrak{n}^3} t^2 \oplus \dots$$

We can write this ring in terms of generators and relations as follows. If $\mathfrak{f} = (f_1, \dots, f_k)$ is the defining ideal of $\mathbb{V}$ then we set the initial ideal $\mathfrak{in}(\mathfrak{f})$ to be the ideal generated by all lowest degree terms of elements in f.

Lemma 4.6. *$\tilde{R}/(X_1, \dots, X_n) \cong \mathbb{C}[X_1, \dots, X_n]/\mathfrak{in}(\mathfrak{f})$*

Proof. Let $\mathfrak{m} = (X_1, \dots, X_n) \lhd \mathbb{C}[X_1, \dots, X_n]$ then

$$\tilde{R}/(X_1, \dots, X_n) \cong \mathbb{C} \oplus \frac{\mathfrak{m} + \mathfrak{f}}{\mathfrak{m}^2 + \mathfrak{f}} t \oplus \frac{\mathfrak{m}^2 + \mathfrak{f}}{\mathfrak{m}^3 + \mathfrak{f}} t^2 \oplus \cdots .$$

Construct a morphism

$$\phi : \mathbb{C}[X_1, \dots, X_n] \to \tilde{R}/(X_1, \dots, X_n) : X_i \to X_i t$$

It is easy to check that this morphism is surjective.

If $f \in \mathfrak{f}$ and f_{in} is its lowest degree term (with degree ℓ) then

$$\phi(f_{in}) = f(X_1, \dots, X_n) t^\ell \mod \mathfrak{m}^{\ell+1} t^\ell,$$

which is zero in $\frac{\mathfrak{m}^\ell + \mathfrak{f}}{\mathfrak{m}^{\ell+1} + \mathfrak{f}} t^\ell$. So $\mathfrak{in}(\mathfrak{f}) \subset \mathsf{Ker}\phi$.

On the other hand the morphism ϕ is graded if we give X_i in $\mathbb{C}[X_1, \dots, X_n]$ degree one. This implies that the ideal $\mathsf{Ker}\phi$ is generated by homogeneous elements. If $g \in \mathsf{Ker}\phi$ is homogeneous then $g(X_i) t^\ell = 0 \mod (\mathfrak{m}^{\ell+1} + \mathfrak{f}) t^\ell$ so there is an $h \in \mathfrak{m}^{\ell+1}$ such that $g + h \in \mathfrak{f}$ so $g \in \mathfrak{in}(\mathfrak{f})$. □

The affine variety corresponding to the ring $\mathbb{C}[X_1, \dots, X_n]/\mathfrak{in}(\mathfrak{f})$ is sometimes called the tangent cone, because it is the cone that approximates $\mathbb{V}(R)$ around the zero. So in words the previous lemma says that the exceptional fiber at the zero, corresponds to the points at infinity of the tangent cone.

Up to now we have only blown up affine varieties but one can also do this with other varieties. If $\mathbb{V}$ is any variety, covered by affine varieties $\mathbb{V}_i$, and $\mathbb{U}$ is a closed subvariety of $\mathbb{V}$ then we have a closed subvariety $\mathbb{U}_i = \mathbb{U} \cap \mathbb{V}_i$. This gives an ideal $\mathfrak{u}_i \subset \mathbb{C}[\mathbb{V}_i]$ such that $\mathbb{C}[\mathbb{U}_i] = \mathbb{C}[\mathbb{V}_i]/\mathfrak{u}_i$. For each $\mathbb{V}_i$ we can construct the blow-up $\tilde{\mathbb{V}}_i = \mathbb{P}\mathbb{C}[\mathbb{V}_i] \oplus \mathfrak{u}_i t \oplus \dots$ with projection map $\pi : \tilde{\mathbb{V}}_i \to \mathbb{V}_i$. On the overlap $\mathbb{V}_{ij} := \mathbb{V}_i \cap \mathbb{V}_j$ we also get an ideal $\mathfrak{u}_{ij} \subset \mathbb{C}[\mathbb{V}_{ij}]$ which we can blow-up. The inclusion $\mathbb{V}_{ij} \subset \mathbb{V}_i$ gives a natural embedding $\mathbb{C}[\mathbb{V}_i] \subset \mathbb{C}[\mathbb{V}_{ij}]$ such that $\mathbb{C}[\mathbb{V}_{ij}]\mathfrak{u}_i = \mathfrak{u}_{ij}$ and hence there is an embedding

$$\mathbb{C}[\mathbb{V}_i] \oplus \mathfrak{u}_i t \oplus \dots \subset \mathbb{C}[\mathbb{V}_{ij}] \oplus \mathfrak{u}_{ij} t \oplus \dots$$

which gives a map $\iota_{ij} : \tilde{\mathbb{V}}_{ij} \to \tilde{\mathbb{V}}_i$. Using these maps we can glue the blow-ups together to the blow-up of $\mathbb{V}$ at $\mathbb{U}$:

$$\tilde{\mathbb{V}} = \bigcup_i \tilde{\mathbb{V}}_i / \sim$$

where $x_i \sim x_j$ for $x_i \in \tilde{\mathbb{V}}_i$ if $\iota_{ij} x_i = \iota_{ji} x_j$.

We now arrive at the most amazing theorem in the course.

Theorem 4.7 (Hironaka). *Every (affine) variety $\mathbb{X}$ can be resolved by a sequence of blow-ups: there is a sequence of maps*

$$\tilde{\mathbb{X}} = \mathbb{X}_n \to \mathbb{X}_{n-1} \to \dots \to \mathbb{X}_1 \to \mathbb{X}$$

such that $\tilde{\mathbb{X}}$ is smooth and each map $\mathbb{X}_{i+1} \to \mathbb{X}_i$ is a blow-up at some closed subvariety of $\mathbb{X}_i$

Proof. The proof of this theorem is very long and technical and not at all obvious. It earned Hironaka a Fields medal in 1970. □

This sequence is not unique, there may be several ways to blow-up the singularity. Moreover the resolution itself is not unique either one singularity can have many different resolutions. In the next section we will show how this works in the case of the Kleinian singularities.

4.5 Blowing up Kleinian singularities

We will now calculate the blow-ups corresponding to the Kleinian singularities of type A_n. The other types will be treated in the appendices.

The ring $\mathbb{C}[X,Y,Z]/(XY-Z^2)$ has a unique singularity in the point $(0,0,0)$ because

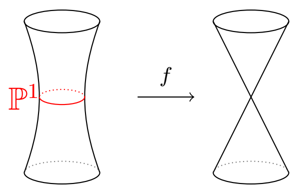
$$(\partial_X,\partial_Y,\partial_Z)r = (Y,X,2Z) = 0 \Leftrightarrow (X,Y,Z) = (0,0,0).$$

The blow-up is (using the convention $x = Xt, y = Yt, z = Zt$)

$$\begin{aligned}
&\mathbb{P}\mathbb{C}[X,Y,Z]/(r) \oplus (X,Y,Z)t \oplus (X,Y,Z)^2t^2 \oplus \cdots \\
&= \mathbb{P}\frac{\mathbb{C}[X,Y,Z,x,y,z]}{(XY-Z^2, Xy-xY, xZ-Xz, Yz-yZ, Xy-Zz, xy-z^2)} \\
&= \{(X,Y,Z,X:Y:Z) \in \mathbb{C}^3\setminus\{0\}\times\mathbb{P}^2 | XY-Z^2=0\} \\
&\quad \cup \{(0,0,0,x:y:z) \in \{0\}\times\mathbb{P}^2 | xy-z^2=0\}
\end{aligned}$$

where the last bit is the exceptional fiber, it is a conic and hence as a variety it is isomorphic to $\mathbb{P}^1$. We can cover the blow-up variety by two parts corresponding to

- $x \neq 0$ we can choose coordinates $X, \eta = y/x, \zeta = z/x$ the relation $xy - z^2 = 0$ becomes $(\eta - \zeta^2) = 0$. which is smooth.
- $y \neq 0$ we can choose coordinates $\xi = x/y, Y, \zeta = z/y$ the relation $xy - z^2 = 0$ becomes $(\xi - \zeta^2) = 0$. which is smooth.
- $z \neq 0$ is not necessary because it implies that both $x, y \neq 0$.



The ring $\mathbb{C}[X,Y,Z]/(XY-Z^n), n \geq 3$ has a unique singularity in the point $(0,0,0)$ because

$$(\partial_X,\partial_Y,\partial_Z)r = (Y,X,3Z^2) = 0 \Leftrightarrow (X,Y,Z) = (0,0,0).$$

The blow-up is

$$\begin{aligned}
&\mathbb{P}\frac{\mathbb{C}[X,Y,Z,x,y,z]}{(XY-Z^n, Xy-xY, \ldots, Xy-Z^{n-1}z, xy-Z^{n-2}z^2)} \\
=&\{(X,Y,Z,X,Y,Z) \in \mathbb{C}^3\setminus\{0\}\times\mathbb{P}^2 | XY-Z^n\} \\
&\cup \{(0,0,0,x,y,z) \in \{0\}\times\mathbb{P}^2 | xy=0\}
\end{aligned}$$

where the last bit is the exceptional fiber, it is a union of 2 projective lines that intersect in the point $(0,0,0,0,0,1)$.

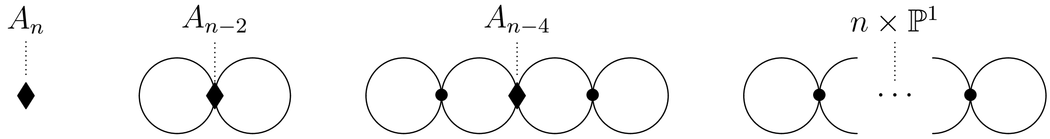
We can cover the blow-up variety by three parts corresponding to

$x \neq 0$ we can choose coordinates $X, \eta = y/x, \zeta = z/x$ the relation $xy - Z^{n-2}z^2 = 0$ gives $\eta - \zeta^n X^{n-2} = 0$. which is smooth.

$y \neq 0$ we can choose coordinates $\xi = x/y, Y, \zeta = z/y$ the relation $xy - Z^{n-2}z^2 = 0$ gives $\xi - \zeta^2 Y^{n-2} = 0$. which is smooth.

$z \neq 0$ we can choose coordinates $\xi = x/z, \eta = y/z, Z$ the relation $xy - Z^{n-2}z^2$ gives $\xi\eta - Z^{n-2} = 0$, which has a singularity if $n > 3$, but this singularity is 'smaller' so we can blow it up again.

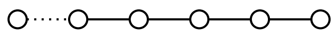
Diagramatically we get the following

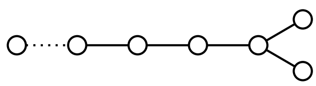


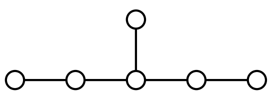
In this picture each circle represents a $\mathbb{P}^1$, which is topologically a sphere. The intersection points intersection points between the spheres are normal. This means that the tangent spaces of the two spheres only intersect in the zero.

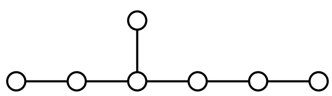
It is custom to represent the exceptional fiber by a graph: each projective line becomes a node and we draw an edge between two nodes if and only if the two projective lines intersect. These graphs can be used to describe all exceptional fibers for the Kleinian singularities.

Theorem 4.8. *If we resolve the Kleinian singularities by consecutive blow-ups we get a resolution $\tilde{\mathbb{X}} \to \mathbb{X}$. The exceptional fiber of the zero is a union of projective lines that intersect normally. If we represent each projective line by a node and draw an edge between two lines that intersect we get the following diagrams:*

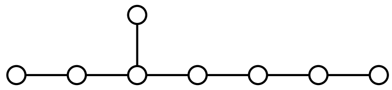
A_n 

D_n 

E_6 

E_7 

E_8



These diagrams are called the simply laced Dynkin diagrams. *The number of nodes is called the rank of the Dynkin diagram and is equal to the subscripted number in the type.*

5

Quivers and quotients

In this chapter we will develop the technology of quivers, which will be very useful in the construction of resolutions of singularities.

5.1 Representations of quivers

Suppose we have an n-dimensional representation of the path algebra. This is a map $\rho : \mathbb{C}Q \to \mathsf{End}_{\mathbb{C}}(W)$ where W is an n-dimensional vector space.

We can decompose the vector space $W = \mathbb{C}^n$ into a direct sum.

$$W \cong \rho(v_1)W \oplus \cdots \oplus \rho(v_k)W.$$

Note that $\rho(v_i)$ acts like the identity on $\rho(v_i)W$ because v_i is an idempotent. Choosing bases in $\rho(v_i)W$ we can associate with every arrow a of Q a matrix W_a corresponding to the map

$$\rho(a)|_{\rho(t(a))W} : \rho(t(a))W \to \rho(h(a))W.$$

This motivates the following definitions: A *dimension vector* of a quiver is a map $\alpha : Q_0 \to \mathbb{N}$, the size of a dimension vector is defined as $|\alpha| := \sum_{v \in Q_0} \alpha_v$. A couple (Q, α) consisting of a quiver and a dimension vector is called a *quiver setting* and for every vertex $v \in Q_0$, α_v is referred to as the dimension of v.

An *α-dimensional complex representation* W of Q assigns to each vertex v a linear space $\mathbb{C}^{\alpha_v}$ and to each arrow a a matrix

$$W_a \in \mathsf{Mat}_{\alpha_{h(a)} \times \alpha_{t(a)}}(\mathbb{C})$$

The space of all α-dimensional representations is denoted by $\mathsf{Rep}(Q, \alpha)$.

$$\mathsf{Rep}(Q, \alpha) := \bigoplus_{a \in Q_1} \mathsf{Mat}_{\alpha_{h(a)} \times \alpha_{t(a)}}(\mathbb{C})$$

If W is a representation of Q and $p = a_1 \dots a_k$ is a path we can define $W_p = W_{a_1} \dots W_{a_k}$ and if $q = \lambda_1 p_1 + \dots + \lambda_n p_n$ is a linear combination of paths with the same head and tail we set $W_q = \lambda_1 W_{p_1} + \dots + \lambda_n W_{p_n}$.

To the dimension vector α we can also assign a group

$$\mathsf{GL}_\alpha := \bigoplus_{v \in Q_0} \mathsf{GL}_{\alpha_\mathsf{v}}(\mathbb{C}).$$

An element of this group, g, has a natural action on $\mathsf{Rep}(Q, \alpha)$:

$$W := (W_a)_{a \in Q_1}, \; W^g := (g_{h(a)} W_a g^{-1}_{t(a)})_{a \in Q_1}$$

and representations in the same orbit represent isomorphic $\mathbb{C}Q$-representations. If $W \in \mathsf{Rep}(Q, \alpha)$ we denote the corresponding $\mathbb{C}Q$-representation by ρ_W, but sometimes we will also write sloppily $\rho \in \mathsf{Rep}(Q, \alpha)$ to denote a $\mathbb{C}Q$-representation.

If A is an algebra of the form $\mathbb{C}Q/\langle r_i | r_i \in \mathcal{R}\rangle$ we can split each relation in its components by multiplying both sides with vertex idempotents. Hence we can assume that every path in the relation r_i has the same head and tail. A representation of Q will correspond to a representation of A if $W_{r_i} = 0$ for all relations r_i. Therefore we can define a closed subvariety $\mathsf{Rep}(A, \alpha) \subset \mathsf{Rep}(Q, \alpha)$ that contains all representations of Q that can be seen as representations of A. $\mathsf{Rep}(A, \alpha)$ will be closed under the action of GL_α and the orbits will be isomorphism classes of A-representations.

5.2 Mumford quotients

Just like in the case of finite groups we want to construct a quotient for this action, but now new problems arise because the group GL_α is not finite. We illustrate this with an example.

Example 5.1. Consider the quiver setting $\textcircled{1} \leftleftarrows \textcircled{1}$. In this case $\mathsf{Rep}(Q, \alpha) = \mathbb{C}^2$ and $\mathsf{GL}_\alpha = \mathbb{C}^{*2}$ which acts by $(\lambda, \mu)(x, y) \mapsto (\lambda\mu^{-1}x, \lambda\mu^{-1}y)$. The orbits of this action come in two types. First there is the origin, which forms an orbit by itself and then there is an orbit for each $(x : y) \in \mathbb{P}^2$. If we take the ring of invariants we only get

$$\mathbb{C}[X, Y]^{\mathsf{GL}_\alpha} = \mathbb{C},$$

so the categorical quotient is just one point. To get the $\mathbb{P}^1$ orbits we need to construct $\mathsf{Proj}\ \mathbb{C}[X, Y]$.

In this section we will construct a new type of quotient which can see more orbits than the categorical quotient.

A *character* of a group G is a group morphism $\theta : \mathsf{G} \to \mathbb{C}^*$. If θ is a character, we write $n\theta$ for the character $\theta : \mathsf{G} \to \mathbb{C}^* : g \mapsto \theta(g)^n$ and the zero character corresponds to the trivial morphism $0 : G \mapsto \{1\} \subset \mathbb{C}^*$. If $\mathbb{X}$ is a variety with a G-action then we say that $f \in \mathbb{C}[\mathbb{X}]$ is a θ-semi-invariant if $f(g \cdot x) = \theta(g) f(x)$. The space of θ-semi-invariants is denoted by $\mathbb{C}[\mathbb{X}]_\theta$. This space does not form a ring because the product of two θ-semi-invariants is a 2θ-semi-invariant, but we can make a graded ring by putting all $n\theta$-semi-invariants together.

$$\mathsf{SI}(\mathbb{X}, \theta) = \bigoplus_{n \geq 0} \mathbb{C}[\mathbb{X}]_{n\theta}.$$

Because this ring is graded we can look at its proj and this variety is called the Mumford quotient:

$$\mathbb{X} /\!\!/_\theta \mathsf{G} := \mathsf{Proj}\ \mathsf{SI}(\mathbb{X}, \theta).$$

Now let $f_1, \dots, f_t, s_1, \dots s_u$ be generators for the ring $\mathsf{SI}(\mathbb{X}, \theta)$ where the f_i have degree 0 (i.e. they are invariants) and the s_i have degree > 0. For each s_i we can construct the ring

$$\mathsf{SI}(\mathbb{X}, \theta)[s_i^{-1}]_0 = (\mathbb{C}[\mathbb{X}][s_i^{-1}])^{\mathsf{G}},$$

which is the categorical quotient $\left(\mathbb{X} \setminus s_i^{-1}(0)\right) /\!\!/ \mathsf{G}$. This means that the Mumford quotient can be covered by categorical quotients of affine open subsets of $\mathbb{X}$.

A point $p \in \mathbb{X}$ is called GIT-θ-semistable (or just semistable[1]) if there is an $n\theta$-semi-invariant function that is nonzero for p. Every semistable point will be nonzero for at least one of the s_i so the Mumford quotient can be seen as a kind of categorical quotient for the semistable points.

For the group GL_α the characters are given by vectors $\theta \in \mathbb{Z}^{Q_0}$:

$$\theta : \mathsf{GL}_\alpha \to \mathbb{C}^* : (g_v)_{v \in Q_0} \mapsto \prod_{v \in Q_0} (\det g_v)^{\theta_v}.$$

[1]GIT stands for geometric invariant theory

Definition 5.2. We define the moduli space of θ-semistable representations of Q as

$$\mathcal{M}_\theta(Q,\alpha) = \mathsf{Rep}(Q,\alpha) /\!\!/_\theta \mathsf{GL}_\alpha,$$

and similarly for an algebra $A = \mathbb{C}Q/\langle r_i \rangle$ we set

$$\mathcal{M}_\theta(A,\alpha) = \mathsf{Rep}(A,\alpha) /\!\!/_\theta \mathsf{GL}_\alpha.$$

Lemma 5.3. *If* $\alpha \cdot \theta := \sum_{v \in Q_0} \alpha_v \theta_v \neq 0$ *then*

$$\mathcal{M}_\theta(Q,\alpha) = \emptyset$$

Proof. One can check that there are no θ-semi-invariants because a scalar matrix $\lambda \in \mathsf{GL}_\alpha$ will act trivially on $\mathsf{Rep}(Q,\alpha)$ but as $\lambda^{\alpha\cdot\theta}$ on a θ-semi-invariant. $\square$

Example 5.4. If we return to the example 5.1 we see that for $\textcircled{1} \Longleftarrow \textcircled{1}$ the character must be of the form $(\theta_1, -\theta_1)$. A semi-invariant is a function $f \in \mathbb{C}[X,Y]$ such that

$$f(\lambda\mu^{-1}x, \lambda\mu^{-1}y) = \lambda^{\theta_1}\mu^{-\theta_1} f(x,y),$$

so it is a function of degree θ_1. Depending on the θ_1 there are 3 cases.

- If $\theta_1 < 0$ the ring of semi-invariants is just $\mathbb{C}$ and quotient is empty.
- If $\theta_1 = 0$ the ring of semi-invariants is $\mathbb{C} \oplus \mathbb{C} \oplus \ldots \cong \mathbb{C}[t]$ and quotient is just a point.
- If $\theta_1 > 0$ the ring of semi-invariants is $\mathbb{C}[X^{\theta_1}, X^{\theta_1 - 1}Y, \ldots, Y^{\theta_1}]$. The quotient is a $\mathbb{P}^1$ (use the map $(x:y) \mapsto (x^{\theta_1} : \cdots : y^{\theta_1})$).

5.3 Finding semi-invariants

In general we can ask the question, how to find invariants and semi-invariants for quiver representations.

Given a quiver Q a cycle is a path p with $h(p) = t(p)$. If W is a representation of Q and p is a cyclic path then W_p is a square matrix that changes by conjugation $g \cdot W_p = g_{h(p)} W_p g_{h(p)}^{-1}$. This means that the trace $\operatorname{Tr} W_p$ is invariant under the GL_α-action and the map

$$t_p : \mathsf{Rep}(Q,\alpha) \to \mathbb{C} : W \mapsto \operatorname{Tr} W_p$$

is an element of $\mathbb{C}[\mathsf{Rep}(Q,\alpha)]^{\mathsf{GL}_\alpha}$.

Theorem 5.5 (Le Bruyn-Procesi). *The ring $\mathbb{C}[\mathsf{Rep}(Q,\alpha)]^{\mathsf{GL}_\alpha}$ is generated by all function of the form t_p for p a cyclic path in Q.*

Note that there are an infinite number of such functions but you only need a finite number of them because $\mathbb{C}[\mathsf{Rep}(Q,\alpha)]^{\mathsf{GL}_\alpha}$ is a finitely generated ring.

Example 5.6. If we take a quiver with one vertex and one loop and $\alpha = 2$, the space of representations is just the space of 2×2-matrices with the conjugation action of $\mathsf{GL}_n(\mathbb{C})$. If we call this matrix X, the invariants are

$$\operatorname{Tr} X, \operatorname{Tr} X^2$$

Other invariants can be expressed in terms of these: $\det X = \frac{1}{2}((\operatorname{Tr} X)^2 - \operatorname{Tr} X^2)$. Using the Caley-Hamilton identity $X^2 - \operatorname{Tr}(X)X + \det X = 0$ one can express $\operatorname{Tr} X^3 = \operatorname{Tr}(X)\operatorname{Tr}(X^2) - \det X \operatorname{Tr} X$.

Example 5.7. If Q is a quiver and $\alpha = (1, \ldots, 1)$ then we do not need to take traces, the invariant is just the product of the values of all its arrows. If a cycle runs through a given vertex twice its invariant is the product of the two smaller cycles. E.g. for

$$\begin{tikzcd} \textcircled{1} \arrow[r, shift left, "a_1,\ldots,a_k"] & \textcircled{1} \arrow[l, shift left, "b_1,\ldots,b_k"] \end{tikzcd}$$

There are kl invariants $t_{ij} = a_i b_j$ which generate all other invariants.

A way to construct a θ-semi-invariant is the following: let $i_1, \ldots, i_s$ be the vertices for which θ_{i_ℓ} is negative, while $j_1, \ldots, j_t$ be the ones with a positive θ_{j_ℓ}. Now choose for each i and j $|\theta_i\theta_j|$ elements in $j\mathbb{C}Qi$ and put all these in a $\sum_j |\theta_j| \times \sum_i |\theta_i|$-matrix D over $\mathbb{C}Q$.

$$D := \begin{bmatrix} j_1\leftarrow i_1 & \cdots & j_1\leftarrow i_1 & & j_1\leftarrow i_s & \cdots & j_1\leftarrow i_s \\ \vdots & |\theta_{j_1}\theta_{i_1}|\times & \vdots & \cdots & \vdots & |\theta_{j_1}\theta_{i_s}|\times & \vdots \\ j_1\leftarrow i_1 & \cdots & j_1\leftarrow i_1 & & j_1\leftarrow i_s & \cdots & j_1\leftarrow i_s \\ \vdots & & & \ddots & & & \vdots \\ j_t\leftarrow i_1 & \cdots & j_t\leftarrow i_1 & & j_t\leftarrow i_s & \cdots & j_t\leftarrow i_s \\ \vdots & |\theta_{j_t}\theta_{i_1}|\times & \vdots & \cdots & \vdots & |\theta_{j_t}\theta_{i_s}|\times & \vdots \\ j_t\leftarrow i_1 & \cdots & j_t\leftarrow i_1 & & j_t\leftarrow i_s & \cdots & j_t\leftarrow i_s \end{bmatrix}$$

If $W \in \mathsf{Rep}(Q,\alpha)$ then we can substitute each entry in D to its corresponding matrix-value in W. In this way we obtain a block matrix D_W with dimensions

$\sum_i \alpha_i|\theta_i| \times \sum_j \alpha_j|\theta_j|$. One can easily check that

$$D_{g\cdot W} = \begin{bmatrix} g_{j_1} & & & & \\ & \ddots & & & \\ & & g_{j_1} & & & \\ & & & \ddots & & \\ & & & & g_{j_t} & \\ & & & & & \ddots & \\ & & & & & & g_{j_t} \end{bmatrix} D_W \begin{bmatrix} g_{i_1}^{-1} & & & & \\ & \ddots & & & \\ & & g_{i_1}^{-1} & & & \\ & & & \ddots & & \\ & & & & g_{i_s}^{-1} & \\ & & & & & \ddots & \\ & & & & & & g_{i_s}^{-1} \end{bmatrix}$$

So if D_W is a square matrix the determinant of D_W is a θ-semi-invariant:

$$\det D_{g\cdot W} = \det g_{j_1}^{|\theta_{j_1}|} \cdots \det g_{j_t}^{|\theta_{j_t}|} \det D_W \det g_{i_1}^{-|\theta_{i_1}|} \cdots \det g_{i_s}^{-|\theta_{i_s}|} = g^\theta \det D_W.$$

We will call these semi-invariants *determinantal semi-invariants.*

Theorem 5.8 (Schofield-Van den Bergh)**.** *As a $\mathbb{C}[\mathsf{Rep}_\alpha Q]^{\mathsf{GL}_\alpha}$-module $\mathbb{C}[\mathsf{Rep}_\alpha Q]_\theta$ is generated by determinantal semi-invariants. As a ring $\mathsf{SI}_\theta[\mathsf{Rep}_\alpha Q]$ is generated by invariants (i.e. traces of cycles) and determinantal $n\theta$-semi-invariants with $n \in \mathbb{N}$.*

Note that this implies that there are only θ-semi-invariants if D_W is a square matrix so $\sum_i \alpha_i|\theta_i| = \sum_j \alpha_j|\theta_j|$ or equivalently $\theta \cdot \alpha = 0$.

Example 5.9. For the quiver setting below with $\theta = (-2, 1)$

$$\textcircled{2} \xRightarrow{a_1,\ldots,a_k} \textcircled{1}$$

a determinantal θ-semi-invariant is given by

$$\det \begin{pmatrix} \sum_i \lambda_i \rho(a_i) \\ \sum_j \mu_j \rho(a_j) \end{pmatrix} = \sum_{ij} \lambda_i \mu_j \det \begin{pmatrix} \rho(a_i) \\ \rho(a_j) \end{pmatrix} = \sum_{ij} \lambda_i \mu_j (\rho(a_i)_1 \rho(a_j)_2 - \rho(a_i)_2 \rho(a_j)_1).$$

where we have used the row linearity of the determinant and the fact that $\rho(a_i)$ is a row vector. For the same reasons a 2θ-semi-invariant is given by linear combinations of

$$\det \begin{pmatrix} \rho(a_i) & \rho(a_j) \\ \rho(a_k) & \rho(a_l) \end{pmatrix}.$$

5.4 Stability in Representation theory

In this section we will describe a criterion that one can use to check whether a representation is semi-stable or not.

Recall that a subrepresentation of a representation $\rho : A \to \mathsf{End}(V)$ is a subspace $W \subset V$ such that is invariant under A: $\rho(A)W = W$. The dimension vector of a subrepresentation W is the map $\alpha_W : Q_0 \to \mathbb{N} : v \mapsto \dim \rho_v W$.

Let $\theta : Q_0 \to \mathbb{Z}$ be the vector corresponding to a character of GL_α. A representation $\rho : A \to \mathsf{GL}(V)$ is called

- RT-θ-semistable if $\alpha_V \cdot \theta = 0$ and $\alpha_W \cdot \theta \geq 0$ for all proper subrepresentations W,
- RT-θ-stable if $\alpha_V \cdot \theta = 0$ and $\alpha_W \cdot \theta > 0$ for all proper subrepresentations W,
- RT-θ-polystable if it is the direct sum of RT-θ-stable representations.

Theorem 5.10. *If $\rho \in \mathsf{Rep}(Q, \alpha)$ is*

- *RT-θ-semistable iff GIT-θ-semistable.*
- *RT-θ-stable iff it is GIT-θ-semistable and the stabilizer of ρ are the scalar matrices $\mathbb{C}^* \subset \mathsf{GL}_\alpha$.*
- *RT-θ-polystable iff it is GIT-θ-semistable and the orbit of ρ is closed in $\mathsf{Rep}^{\theta-ss}(Q, \alpha)$.*

Proof. The proof can be found in [**?**][Chapter 4]. □

As GIT-semistability and RT-semistability coincide, we will use the terms θ-semistable, θ-stable and θ-polystable.

Note that if $\theta = 0$ then every representation is θ-semistable, θ-stable is the same as simple and θ-polystable is the same as semisimple.

5.5 Stability and moment maps

We have seen that we can construct $\mathcal{M}_\theta(Q, \alpha)$ as the proj of a graded ring. In this view $\mathcal{M}_\theta(Q, \alpha)$ classifies all closed orbits in the space of semistable representations. Not all semistable representations have closed orbits, only the polystable representations. Therefore $\mathcal{M}_\theta(Q, \alpha)$ is not a set-theoretical quotient but a categorical quotient.

One of the problems in the construction is to check whether a representation is polystable or not. To do this we must check whether there is a nonzero semi-invariant. After that we still have to check whether the orbit is closed or not.

Another approach is to find equations for a subspace of $\mathsf{Rep}(Q,\alpha)$ that contains only polystable representations and that meets every polystable orbit in at least one point.

Theorem 5.11 (King)**.** *Let Q be a quiver, α a dimension vector and θ a character. A representation ρ is θ-polystable if and only if there is a $\rho' \in \mathsf{GL}_\alpha \cdot \rho$ such that*

$$\forall v \in Q_0 : \sum_{h(a)=v} \rho'(a)\rho'(a)^\dagger - \sum_{t(a)=v} \rho'(a)^\dagger\rho'(a) = i\theta_v 1_{\alpha_v}.$$

Here $\dagger$ stands for the hermitian transpose.

Proof. The proof can be found in [**?**]. $\square$

If we define *the moment map*

$$\mu_{\mathbb{R}} : \mathsf{Rep}(Q,\alpha) \to \mathfrak{gl}_\alpha = \prod_{v\in Q_0} \mathtt{mat}_{\alpha(v)\times\alpha(v)}(\mathbb{C}) : \rho \mapsto \sum_{h(a)=v} \rho(a)\rho(a)^\dagger - \sum_{t(a)=v} \rho(a)^\dagger\rho(a)$$

and $\vec{\theta} = (i\theta_v 1_{\alpha_v})_{v\in Q_0} \in \mathfrak{gl}_\alpha$ then it is clear that $\mu_{\mathbb{R}}^{-1}(\vec{\theta})$ is a space that meets every closed semistable orbit.

If $\rho \in \mu_{\mathbb{R}}^{-1}(\vec{\theta})$ then $g\rho$ does not necessarily sit in $\mu_{\mathbb{R}}^{-1}(\vec{\theta})$ but if we restrict to the case

$$gg^\dagger := (g_v g_v^\dagger)_{v\in Q_0} = (1_v)_{v\in Q_0}$$

then this is the case. So there is an action of the group $\mathsf{U}_\alpha := \{g \in \mathsf{GL}_\alpha | gg^\dagger = 1\}$ on the space $\mu_{\mathbb{R}}^{-1}(\vec{\theta})$.

Theorem 5.12 (King)**.** *The embedding $\mu_{\mathbb{R}}^{-1}(\vec{\theta}) \subset \mathsf{Rep}(Q,\alpha)$ induces a homeomorphism between*

$$\mu_{\mathbb{R}}^{-1}(\vec{\theta})/U_\alpha \text{ and } \mathcal{M}_\theta(Q,\alpha)$$

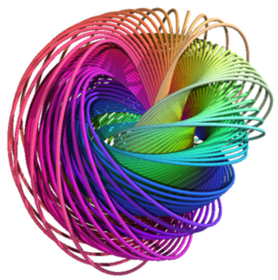
Proof. This follows immediately from theorem 5.11 $\square$

Note that this map is just a homeomorphism between topological spaces, not an isomorphism of varieties because the source does not have the structure of a complex variety as the map $\mu_{\mathbb{R}}$ is not holomorphic.

We illustrate the theorem with a small example. If we want to construct $\mathbb{P}^1$ we can do this via the moduli space $\mathcal{M}_{(1,-1)}(\textcircled{1} \Leftarrow \textcircled{1})$. In this case

$$\mu_{\mathbb{R}}^{-1}(1,-1) = \{(a,b) \in \mathbb{C}^2 | aa^\dagger + bb^\dagger = 1, -a^\dagger a - b^\dagger b = -1\}$$

This is just the real unit 3-sphere embedded in $\mathbb{R}^4 = \mathbb{C}^2$. The group $\mathsf{U}_\alpha = \mathsf{U}_1 \times \mathsf{U}_1$ acts on the 3-sphere by $(\lambda, \mu)(a,b) = (\lambda\mu^{-1}a, \lambda\mu^{-1}b)$. All orbits are circles and the quotient $\mu_{\mathbb{R}}^{-1}(1,-1) \to \mu_{\mathbb{R}}^{-1}(1,-1)/\mathsf{U}_\alpha$ is a circle bundle over the sphere. This bundle is known as the Hopf-fibration. Below is a picture of the Hopf-fibration where the 3-sphere is projected stereographically to $\mathbb{R}^3$ plus a point at infinity. The circles are all closed curves except the vertical line, which corresponds to the orbit of the point at infinity.



6
Braiding it all together

In this chapter we will use preprojective algebras to study the deformations and resolutions of Kleinian singularities.

6.1 Resolutions of Kleinian singularities

As we already know the preprojective algebra of an extended Dynkin quiver is Morita equivalent to the skew group ring associated to a Kleinian singularity.

$$\Pi(Q) = e\mathbb{C}[V] \star \mathsf{G}e$$

We also constructed a dimension vector which assigned to each vertex the dimension of the corresponding simple representation. In this section we denote this dimension vector by ς.

Theorem 6.1. $\mathbb{C}[\mathsf{Rep}(\Pi(Q),\varsigma)]^{\mathsf{GL}_\varsigma} \cong \mathbb{C}[V]^{\mathsf{G}}$

Proof. We do the proof in the A_n-case and refer to the appendices for the other cases.

For A_n the coordinate ring of the representation space has the following presentation

$$\mathbb{C}[\mathsf{Rep}(\Pi(Q),\varsigma)] = \mathbb{C}[X_i, Y_i | 0 \leq i \leq n]/(X_iY_i - X_{i+1}Y_{i+1} | 0 \leq i \leq n).$$

A monomial m in the X_i, Y_i will correspond to an invariant if it is a product of cycles in the quiver because each path p transforms as $g_{h(p)}g_{t(p)}^{-1}$. The homology of the quiver (viewed as a CW-complex) is generated by the X_iY_i (which all correspond to the same element in $\zeta \in \mathbb{C}[\mathsf{Rep}(\Pi(Q),\varsigma)]$), $\xi = X_0 \dots X_n$ and $\eta = Y_0 \dots Y_n$. The relation between these 3 invariants is $\xi\eta - \zeta^{n+1}$.

□

Corollary 6.2.

$$\mathcal{M}_0(\Pi,\varsigma) \cong V/\!\!/\mathsf{G}$$

From the construction of the moduli space we know that for any θ the map

$$\pi : \mathcal{M}_\theta(\Pi,\varsigma) \to \mathcal{M}_0(\Pi,\varsigma)$$

is surjective, proper and one to one on an open subset, so because $\mathcal{M}_0(\Pi,\varsigma)$ is an irreducible variety this map is almost everywhere one-to-one and π is at least a partial resolution.

Definition 6.3. A stability condition $\theta : Q_0 \to \mathbb{Z}$ with $\theta \cdot \varsigma = 0$ is called *generic* if there is no dimension vector $\beta \neq \varsigma$ with $\beta_v \leq \varsigma_v$ such that $\beta \cdot \theta = 0$. For a generic stability condition the notions of stability, semistability and polystability coincide.

Theorem 6.4. *If θ is generic then*

$$\pi : \mathcal{M}_\theta(\Pi,\varsigma) \to \mathcal{M}_0(\Pi,\varsigma)$$

is a resolution.

Proof. We do the proof in the A_n-case and refer to the appendices for the other cases.

First we look at the smooth locus of $\mathsf{Rep}(\Pi,\varsigma)$. The dimension of this space is $n+2$ because if all X_i, Y_i are invertible we have $Y_i = X_0Y_0/X_i$, so we there are $2n+2-n = n+2$ variables we can chose freely.

The generating relations are $X_iY_i - X_{i+1}Y_{i+1}$ $(i \leq n-1)$ (we don't need the last one because it is minus the sum of the others). The Jacobian matrix becomes this $n \times 2n+2$ matrix

$$\begin{pmatrix} Y_0 & X_0 & -Y_1 & -X_1 & 0 & & & \dots & 0 \\ 0 & 0 & Y_1 & X_1 & -Y_2 & -X_2 & 0 & \dots & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & \dots & & & & & & -Y_n & X_n \end{pmatrix}$$

The point $(x_i, y_i)_{0\leq i\leq n}$ is smooth if this matrix has rank n. This is precisely when at least n of the $n+1$ pairs (x_i, y_i) are nonzero.

If there is more than one such pair zero then the representation splits as a direct sum $W_1 \oplus W_2$ because the set of vertices splits in 2 subsets with no nonzero arrows between these 2 sets. If θ is generic then $\theta \cdot \alpha_{W_1} \neq 0$ and because $\theta \cdot (\alpha_{W_1} + \alpha_{W_2}) \neq 0$ the dot product of θ with either α_{W_1} or α_{W_2} must be negative, so the representation $W_1 \oplus W_2$ is not stable.

Therefore all θ-stable points are smooth points of $\mathsf{Rep}(\Pi, \varsigma)$. Also all stabilizers of stable points are $\mathbb{C}^*$ so all orbits are diffeomorphic to $\mathsf{GL}_\varsigma/\mathbb{C}^* = \mathsf{PGL}_\varsigma$. So $\mathsf{Rep}^{\theta-ss}(\Pi, \varsigma)$ is a PGL_ς-fiber bundle with base $\mathcal{M}_\theta(\Pi, \varsigma)$. If the base is not smooth, the total space can also not be smooth so $\mathcal{M}_\theta(\Pi, \varsigma)$ must be smooth and $\pi : \mathcal{M}_\theta(\Pi, \varsigma) \to \mathcal{M}_0(\Pi, \varsigma)$ is a resolution.

A case independent proof of this theorem can be found in [**?**]. $\square$

If we look at the special generic character θ that assigns to the vertex of the trivial representation $-|\varsigma| + 1$ and 1 to all other vertices then there is also an easy description of the exceptional fiber.

Theorem 6.5. *For $\theta = (-|\varsigma|+1, 1\ldots, 1)$ the exceptional fiber consists of a union of $\mathbb{P}^1$'s, whose intersection diagram is the corresponding Dynkin diagram.*

Proof. Again we do the A_n case. Note that in this case to be stable there must be a nonzero path from the trivial vertex v_0 to any other vertex v_i. There are only two such basic paths $X_{i-1}\ldots X_0$ and $Y_i \ldots Y_n$. Furthermore if we perform a base change on ρ, the ratio between these 2 paths does not change. Therefore the map

$$\mathcal{M}_\theta(\Pi, \varsigma) \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 : \rho \mapsto ((\rho(X_{i-1}\ldots X_0) : \rho(Y_i \ldots Y_n)))_{1\leq i\leq n}$$

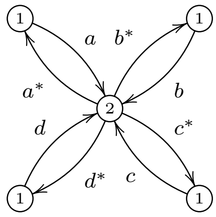
is well-defined. Fix a stable representation ρ and suppose that $(\rho(X_{i-1}\ldots X_0) : \rho(Y_i \ldots Y_n)) \neq (1 : 0)$ then $Y_i \neq 0$ and hence if ρ sits in the exceptional fiber $X_i = 0$ and $(\rho(X_{j-1}\ldots X_0) : \rho(Y_j \ldots Y_n)) = (0 : 1)$ for all $j > i$. Similarly if $(\rho(X_{i-1}\ldots X_0) : \rho(Y_i \ldots Y_n)) \neq (0 : 1)$ then $(\rho(X_{j-1}\ldots X_0) : \rho(Y_j \ldots Y_n)) = (1 : 0)$ for all $j < i$.

From this we can deduce that the image of the exceptional fiber under this map is

$$\bigcup_{i=1\to n} (1:0)^{i-1} \times \mathbb{P}^1 \times (0:1)^{n-i+1},$$

which is precisely a chain of intersecting $\mathbb{P}^1$'s. The map restricted to the exceptional fiber is also an injection because we can make a base change such that all nonzero X_i, Y_j are $0, 1$ except for X_0, Y_n which are determined up to a common multiple. $\square$

Example 6.6. Consider the McKay quiver setting for the D_4 singularity



and take $\theta = (-5, 1, 1, 1, 1)$, where the first vertex (trivial vertex) is the top left one. The preprojective algebra is

$$\mathbb{C}Q/(aa^* + bb^* + cc^* + dd^*, a^*a, b^*b, c^*c, d^*d).$$

Suppose that $\rho \in \mathsf{Rep}(\Pi, \varsigma)$ is a stable representation that sits above the zero point. The latter means that all traces of cycles are zero, while the former implies that there must be a nonzero path from the trivial vertex to all the other vertices. Therefore $\rho(a), \rho(b^*), \rho(c^*), \rho(d^*)$ cannot be zero. For the central vertex we must have that $\rho(a), \rho(b), \rho(c), \rho(d)$ span a 2-dim space.

The kernel of $\rho(a^*)$ contains $\rho(a)$ so if $\rho(a^*)$ were nonzero then either $\rho(a^*b)$, $\rho(a^*c)$, $\rho(a^*d) \neq 0$ but because stability implies there is a nonzero path in the opposite direction one of the cycles would also be nonzero. Therefore $\rho(a^*) = 0$ and

$$\rho((aa^* + bb^* + cc^* + dd^*) = \rho(bb^*) + \rho(cc^*) + \rho(dd^*) = 0$$

This means that at least 2 of the $\rho(b), \rho(c), \rho(d)$ are nonzero and not a multiple of $\rho(a)$. Now we distinguish the following cases

- If $\rho(b) = 0$ we can pick a basis in the middle vertex such that $\rho(a) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, $\rho(c) = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. The relations $\rho(c^*c) = 0$, $\rho(d^*d) = 0$ and $\rho(cc^* + dd^*) = 0$ imply that after base change in the heads of c, d we can assume $\rho(d) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\rho(c^*) = -\rho(d^*) = \left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right)$ The only thing we did not fix is $\rho(b^*)$ which is nonzero and defined up to a multiple by base change in $h(b^*)$. This gives a $\mathbb{P}^1$ of possibilities.
- If $\rho(c) = 0$ this also gives a $\mathbb{P}^1$ of possibilities.
- If $\rho(d) = 0$ this also gives a $\mathbb{P}^1$ of possibilities.
- If $\rho(b), \rho(c), \rho(d) \neq 0$ then these three vectors generate a one-dimensional subspace otherwise we can make a nonzero cycle. Therefore $\rho(bb^*)$, $\rho(cc^*)$, $\rho(dd^*)$ also span a one-dimensional space and The ratio $\rho(bb^*)/\rho(cc^*)$ is invariant under base change. It can be seen as an element in $\mathbb{P}^1 \setminus \{0, \infty, -1\}$ because $\rho(bb^*) + \rho(cc^*) + \rho(dd^*) = 0$.

6.2 From resolving to deforming

In this section we show that the moduli space of θ-semistable representations of a preprojective algebra is homeomorphic to the moduli space of semisimple (=0-semistable) representation of a deformed preprojective algebra.

Theorem 6.7. *Let Π be a preprojective algebra and $\lambda : Q_0 \to \mathbb{Z}$ then there is a homeomorphism between*

$$\mathcal{M}_\lambda(\Pi^0) \text{ and } \mathcal{M}_0(\Pi^\lambda)$$

Proof. For the proof we will use King's criterion for stability in the case of preprojective algebras. Note that the condition for stability

$$\sum_{h(a)=v} \rho(a)\rho(a)^\dagger - \sum_{t(a)=v} \rho(a)^\dagger \rho(a) = \theta_v$$

looks very much like the deformed preprojective relation.

On $\mathsf{Rep}(\bar{Q}, \alpha)$ we can define a $\mathbb{C}$-linear map $\vee$:

$$\rho^\vee(a) = \rho(a^*) \text{ and } \rho^\vee(a^*) = -\rho(a)$$

If we replace the role of $\dagger$ by $\vee$ in $\mu_\mathbb{R}$ we get a new map

$$\mu_\mathbb{C} : \mathsf{Rep}(\bar{Q}, \alpha) \to \mathfrak{gl}_\alpha : \sum_{h(a)=v} \rho(a)\rho(a)^\vee - \sum_{t(a)=v} \rho(a)^\vee \rho(a)$$

and the deformed preprojective relation can be seen as

$$\mu_\mathbb{C}(\rho) = 2\lambda.$$

The factor two comes from the fact that the sum runs over all arrows not just the unstarred ones.

Consider the following bijection $h : \mathsf{Rep}(\bar{Q}, \alpha) \to \mathsf{Rep}(\bar{Q}, \alpha)$

$$h\rho(a) = \frac{1}{\sqrt{2}}\left(i\rho(a) - i\rho(a^*)^\dagger\right) \text{ and } h\rho(a^*) = \frac{1}{\sqrt{2}}\left(i\rho(a^*) + i\rho(a)^\dagger\right)$$

This bijection is U_α-equivariant and

$$\mu_\mathbb{C}(h\rho) = \frac{1}{2}(\mu_\mathbb{C}(\rho)^\dagger - \mu_\mathbb{C}(\rho)) - 2i\mu_\mathbb{R}(\rho) \text{ and } \mu_\mathbb{R}(h\rho) = \frac{i}{2}(\mu_\mathbb{C}(\rho)^\dagger + \mu_\mathbb{C}(\rho)).$$

Now

$$\mathcal{M}_0(\Pi^\lambda, \alpha) = (\mu_\mathbb{C}^{-1}(2\lambda) \cap \mu_\mathbb{R}^{-1}(0))/\mathsf{U}_\alpha \text{ and } \mathcal{M}_\lambda(\Pi, \alpha) = (\mu_\mathbb{C}^{-1}(0) \cap \mu_\mathbb{R}^{-1}(\lambda))/\mathsf{U}_\alpha$$

From this it is easy to see that

$$\mathcal{M}_0(\Pi^{2\lambda}, \alpha) = \mathcal{M}_\lambda(\Pi, \alpha)$$

By scaling the matrices with $\sqrt{r}$ we see that $\mathcal{M}_0(\Pi^\lambda, \alpha) \cong \mathcal{M}_0(\Pi^{r\lambda}, \alpha)$, so this proves theorem 6.7. □

Remark 6.8. Note that the homeomorphism in theorem 6.7 cannot be a morphism of varieties because the former is a quasiprojective variety, while the latter is an affine variety.

If Π is a preprojective algebra that comes from a Kleinian singularity then $\mathcal{M}_0(\Pi)$ is a hypersurface defined by a function $f(X, Y, Z)$ which describes the relation between the 3 invariants that generate the ring of invariants. We can find 3 cycles in the quiver whose traces ξ, η, ζ satisfy $f(\xi, \eta, \zeta)$. These generators also can be interpreted as invariants of $\mathsf{Rep}(\Pi^\lambda, \alpha)$, so for each λ the generators ξ, η, ζ will satisfy a different relation $f_\lambda(X, Y, Z)$ and this will give a deformation of f. It might however be the case that different deformation parameters give the same deformation of f.

Example 6.9. For A_2 the coordinate ring of the representation space has the following presentation

$$\mathbb{C}[\mathsf{Rep}(\Pi^\lambda, \varsigma)] = \mathbb{C}[X_i, Y_i | 0 \le i \le 2]/(X_iY_i - X_{i+1}Y_{i+1} - \lambda_i | 0 \le i \le 2).$$

which is trivial unless $\lambda_0 + \lambda_1 + \lambda_2 = 0$.

Just as in the undeformed case the generators are the X_iY_i, $\xi = X_0 \dots X_2$ and $\eta = Y_0 \dots Y_2$. If we put $\zeta = X_0Y_0$ then $X_1Y_1 = \zeta - \lambda_0$ and $X_2Y_2 = \zeta - \lambda_0 - \lambda_1$. The relation between these 3 invariants is

$$\xi\eta - \zeta(\zeta - \lambda_0)(\zeta - \lambda_0 - \lambda_1) = 0.$$

If we substitute $\zeta \to \zeta - \frac{0+\lambda_0+\lambda_0+\lambda_1}{3}$ and assume that $\lambda_0 + \lambda_1 + \lambda_2 = 0$ we get the following

$$\xi\eta - (\zeta + \frac{\lambda_0 - \lambda_2}{3})(\zeta + \frac{\lambda_1 - \lambda_0}{3})(\zeta + \frac{\lambda_2 - \lambda_1}{3}).$$

This function is invariant under the action of $S_3 = \Sigma(\{0, 1, 2\})$ on $\{\vec{\lambda} \in \mathbb{C}^3 | \lambda_0 + \lambda_1 + \lambda_2 = 0\}$

$$\lambda_i \mapsto (-1)^\sigma \lambda_{\sigma(i)}.$$

If we chose a basis $e_1 = (-1, 1, 0), e_2 = (-1, 0, 1)$ for we get the following matrix expressions for the transpositions

$$(01) \mapsto \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},\ (02) \mapsto \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix},\ (12) \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

So there is a group that acts on the parameter space, while keeping the deformation the same.

6.3 Weyl groups and versal deformations

Definition 6.10. If Γ is a graph we can define a bilinear form on $\mathbb{Z}^{\Gamma_0}$

$$\langle x, y\rangle_\Gamma = 2\sum_{i\in\Gamma_0} x_i y_i - \sum_{i - j\in\Gamma_1} x_i y_j - x_j y_i.$$

And for each node of the graph we define a linear map

$$s_i : \mathbb{Z}^{\Gamma_0} \to \mathbb{Z}^{\Gamma_0} : x \mapsto x - \langle x, e_i\rangle e_i$$

where $e_i \in \mathbb{Z}^{\Gamma_0}$ is the standard basis vector with a 1 on the i^{th} place. The group generated by these reflections is called the *Weyl group* W_Γ.

With this in mind we see that S_3 is the Weyl group of the Dynkin diagram A_2.

Theorem 6.11. *The following are equivalent:*

1. *the bilinear form* $\langle , \rangle_\Gamma$ *is positive definite,*
2. *the Weyl group* W_Γ *is finite,*
3. Γ *is a Dynkin diagram.*

Proof. This is a classic and can be found in every book on Lie groups. □

We will now apply this to our situation.

Lemma 6.12. *If* α *is a dimension vector and* $\lambda \cdot \alpha \neq 0$ *then there are no* α*-dimensional representations of* $\Pi^\lambda(Q)$.

Proof. Let ρ be a representation If we take the trace of $\rho(\sum_{a\in Q_1} aa^* - a^*a - \sum_{v\in Q_0} \lambda_v v)$ we see that the trace of the first sum is zero because it is a commutator. If the dimension vector is α the trace of the second sum is $\lambda \cdot \alpha$, so this must be zero as wel. □

Therefore it makes sense to define

$$\Lambda_\alpha := \{\vec{\lambda} \in \mathbb{C}^{Q_0} | \lambda \cdot \alpha = 0\}$$

If Π is the preprojective algebra of a Kleinian singularity and ς the standard dimension vector then we can construct a basis for Λ_ς:

$$e_i = (-\varsigma_i, 0, \dots, 0, 1, 0, \dots, 0) \text{ with the one on the } i^{th} \text{ spot}$$

The n basis vectors correspond to the nontrivial vertices of the McKay quiver, which also correspond to the nodes of the Dynkin diagram Γ associated with the Kleinian singularity. With this identification we get an action of the Weyl group W_Γ on $\Lambda_\varsigma \cong \mathbb{Z}^\Gamma \otimes \mathbb{C}$.

Theorem 6.13. *If $g \in \mathsf{W}_\Gamma$ then f_λ and $f_{g\lambda}$ define isomorphic hypersurfaces.*

Proof. The proof of A_n is analogous to that of A_2. First we make a substitution $\zeta \to \zeta - \frac{1}{n+1}\sum_{i=1}^n \lambda_0 + \cdots + \lambda_{i-1}$ such that the degree n-term becomes zero. If we choose a new parametrization $\mu_0, \dots, \mu_n$ where

$$\mu_j = \lambda_0 + \cdots + \lambda_{j-1}\frac{1}{n+1}\sum_{i=1}^{n} \lambda_0 + \cdots + \lambda_{i-1}$$

then $\sum \mu_j = 0$, $\mu_{j+1} - \mu_j = \lambda_j$ and the equation becomes

$$\xi\eta - (\zeta - \mu_0) \dots (\zeta - \mu_n).$$

We have an action of $\Sigma(\{0, \dots, n\})$ by permuting the μ_i, which leaves the polynomial invariant. The transposition $(i, i+1)$ will act as

$$(-\sum_{i=1}^{n} \lambda_i, \lambda_1, \dots, \lambda_n) \mapsto (-\sum_{i=1}^{n} \lambda_i, \lambda_1, \dots, \lambda_{i-1} + \lambda_i, -\lambda_i, \lambda_{i+1} + \lambda_i, \dots)$$

which is precisely

$$s_i : \mathbb{Z}^{\Gamma_0} \to \mathbb{Z}^{\Gamma_0} : x \mapsto x - \langle x, e_i \rangle e_i$$

where $\langle , \rangle$ comes from the A_n Dynkin diagram. $\square$

Theorem 6.14. *The quotient $\Lambda_\varsigma /\!\!/ \mathsf{W}$ is isomorphic to $\mathbb{C}^n$ and there is a polynomial versal deformation*

$$F : \mathbb{C}^3 \times \Lambda_\varsigma /\!\!/ \mathsf{W} \to \mathbb{C}$$

with $F(x, \mathsf{W} \cdot \lambda) \cong f_\lambda$.

Proof. If we look at the permutation action of $\mathsf{W} = \Sigma_{n+1}$ on $\mathbb{C}^{n+1}$, the ring of invariant functions is the ring of symmetric functions. It is well known that this ring is generated by the coefficients of the polynomial

$$p(t) = (t - u_0) \dots (t - u_n),$$

so

$$\mathbb{C}[u_0, \dots, u_n]^{\Sigma_{n+1}} = \mathbb{C}[u_0 + \cdots + u_n, \dots, u_0 \dots u_n].$$

which is a polynomial ring in $n+1$ variables. We have seen that, by base change to the μ_i, Λ_ς could be identified with the subspace of $\mathbb{C}^{n+1}$ for which $u_0+\cdots+u_n=0$. So the ring of invariants $\mathbb{C}[\Lambda_\varsigma]^{\mathsf{W}}$ is

$$\mathbb{C}[u_0+\cdots+u_n,\ldots,u_0\ldots u_n]/(u_0+\cdots+u_n)$$

which is a polynomial ring in n variables, so $\Lambda_\varsigma /\!\!/ \mathsf{W} \cong \mathbb{C}^n$.

For A_n we already know that

$$\mathtt{Jac}(XY-Z^{n+1})=\mathbb{C}[X,Y,Z]/(X,Y,(n+1)Z^n)=\mathbb{C}[Z]/(Z^n).$$

and we can chose a basis Z^i, $i=0,\ldots,n-1$ to obtain a versal deformation of $XY-Z^{n+1}$. This versal deformation consists of all $XY-p(Z)$ where $p(Z)$ is a monic polynomial of degree $n+1$ for which the coefficient of Z^n is zero. These are precisely those polynomials of the form

$$p(Z)=(Z-\mu_0)\ldots(Z-\mu_n).$$

Therefore the deformation

$$F(x,\mathsf{W}\cdot\lambda)=XY-(Z-\mu_0)\ldots(Z-\mu_n).$$

is the versal deformation we are looking for. $\square$

6.4 Braid groups and Monodromy

We have seen that we can construct a polynomial versal deformation

$$F:\mathbb{C}^3\times\mathbb{C}^n\to\mathbb{C}$$

where $\mathbb{C}^n=\Lambda_\varsigma /\!\!/ \mathsf{W}$.

Theorem 6.15. *Let ς be the standard dimension vector. For each $v\in Q_0$ we define a hyperplane in Λ_ς*

$$H_v:=\{\lambda\in\Lambda_\varsigma|\lambda_v=0\}$$

Denote the complement of the orbits of all hyperplanes by

$$\Lambda_\varsigma^\circ:=\Lambda_\varsigma\setminus\mathsf{W}\cdot\bigcup_{v\in Q_0}H_v.$$

If $p\in\Lambda_\varsigma^\circ /\!\!/ \mathsf{W}$ then $F_p^{-1}(0)$ is smooth.

Proof. The space
$$XY - (Z - \mu_0)\dots(Z - \mu_n) = 0$$
is singular if $\mu_i = \mu_j$ for some $i \neq j$. The Weyl group permutes the indices so up to the action of W we can assume $j = i + 1$. In the original coordinates this means
$$\lambda_i = 0$$
so $\lambda \in H_i$. □

Because the parameter space Λ is complex, the complement of a hyperplane is still connected. This means that Λ_ς° and hence also $\mathbb{L} := \Lambda_\varsigma^\circ/\mathsf{W}$ is connected.

The space $\mathbb{L}$ comes with a bundle of smooth deformations of the singularity
$$\mathbb{B} := \{(X, Y, Z, \lambda) \in \mathbb{C}^3 \times \mathbb{L} | F(X, Y, Z, \lambda) = 0\} \to \mathbb{L}$$
We denote the fibers of $\mathbb{B}$ by $\mathbb{B}_\lambda$.

Given a path $\gamma : [0, 1] \to \mathbb{L}$ we can try to relate the fibers above $\gamma(0)$ and $\gamma(1)$. In order to do this we need to define a vector field V on $\mathbb{B}$ such that under the projection $\pi : \mathbb{B} \to \mathbb{L}$ we have $d\pi(V_x) = \frac{dp}{dt}|_{t=r}$ if $\gamma(r) = \pi(x)$. This is always possible because the map $(d\pi)_x$ is surjective for every $x \in \mathbb{B}$.

If we have found such a V, we can integrate it to obtain a time-one-flow $\phi : \mathbb{B} \to \mathbb{B}$. Because $d\pi(V_x) = \frac{dp}{dt}|_{t=r}$ a point $x \in \pi^{-1}(\gamma(0))$ will flow to a point $y \in \pi^{-1}(\gamma(1))$. And hence $\phi|_{\pi^{-1}(\gamma(0))}$ will be a homeomorphism between $\pi^{-1}(\gamma(0))$ and $\pi^{-1}(\gamma(1))$. From this we can deduce that all fibers in $\mathbb{B}$ are homeomorphic.

The fibers are however not homeomorphic in a canonical way. Different choices of V_0, V_1 will result in different connecting morphisms but two such morphism will be homotopic because the flows coming from $V_t := (1 - t)V_0 + tV_1$ give a homotopy between the two connecting morphisms. Similarly, if γ_0 and γ_1 are two homotopic paths with homotopy γ_t, we can find vector fields V_t such that the resulting flows give a homotopy between the connecting homeomorphisms.

Because everything is defined up to homotopy we can look at the action of these homeomorphisms on the homology of the fiber.

Theorem 6.16. *The homology of the fiber is*
$$H_\bullet(\mathbb{B}_\lambda, \mathbb{C}) = \mathbb{C} \oplus 0 \oplus \mathbb{C}^n \oplus 0 \oplus 0$$

Proof. Because all fibers are diffeomorphic we can look at the fiber for $\lambda = \vartheta = (-n, 1, \dots, 1)$. In this case the fiber is diffeomorphic to the minimal resolution of the singularity.

The singularity itself is contractible because it is of the form $\mathbb{C}^2/\!\!/\mathsf{G}$ where G acts linearly, so scaling by $r \in \mathbb{R}$ will contract it to a point. Therefore $H_i(\mathbb{C}^2/\mathsf{G}) = (\mathbb{C}, 0, 0, 0, 0)$. Now if we blow up a point, we delete this point and add in a $\mathbb{P}^1$. A $\mathbb{P}^1$ has homology $(\mathbb{C}, 0, \mathbb{C}, 0)$ so using standard homological techniques we can deduce that after n blow ups we get.

$$H_\bullet(\tilde{\mathbb{X}}) = (\mathbb{C}, 0, \mathbb{C}^n, 0, 0)$$

□

So the most interesting part of the homology is the second. We can bundle all the second homology spaces together in one bundle $\pi : \mathcal{H} \to \mathbb{L}$ with $\pi^{-1}(\lambda) = H_2(\mathbb{B}_\lambda, \mathbb{R})$. Each path in $\mathbb{L}$ defines a connecting isomorphism between the two fibers, which only depends on the homotopy class of the path. If U is a contractible open set in $\mathbb{L}$, we can use these for a local trivialization. So $\mathcal{H}$ is a vector bundle.

Definition 6.17. The fundamental group of $\mathbb{L}$ is called the braid group of type ADE_n depending on the Dynkin diagram. We denote it by $\mathsf{Br} = \pi_1(\mathbb{L}, \lambda)$

Theorem 6.18. *The braid group has the following presentation: it is generated by elements σ_i one for every vertex i in the Dynkin diagram. Its relations are*

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

if there is no edge between i and j, and

$$\sigma_j \sigma_i \sigma_j = \sigma_i \sigma_j \sigma_i,$$

if there is an edge between i and j.

Proof. In the A_n case the space $\mathbb{L}$ consists of all unordered n+1-tuples $\{\mu_0, \ldots, \mu_n\}$ whose sum is zero. This is a deformation retract of the space of all unordered $n+1$=tuples. To see this one can use the deformation

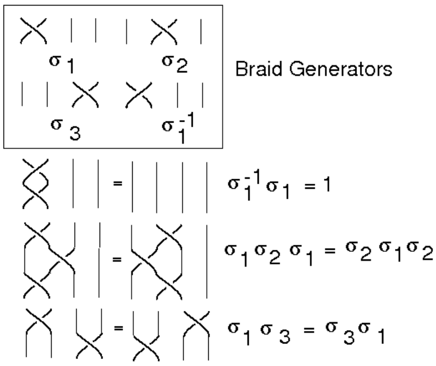
$$\{\mu_0, \ldots, \mu_n\} \to \{\mu_0 - t\frac{\bar{\mu}}{n+1}, \ldots \mu_n - t\frac{\bar{\mu}}{n+1})$$

with $t \in [0, 1]$. So

$$\mathsf{Br} = \pi_1(\{S \subset \mathbb{C} | \#S = n + 1\}, p)$$

If we take as $p = \{0, 1, \ldots, n\}$ a loop γ in $\{S \subset \mathbb{C} | \#S = n + 1\}$ can be seen as a set of points moving in $\mathbb{C}$ starting and ending with p. Using $t \in [0, 1]$ as the third coordinate this can be seen as a braid with $n+1$ strings in $\mathbb{C} \times [0, 1] \subset \mathbb{R}^3$.

The generators and relations can be read off from the following picture.



To show that these are the only relations we refer to [?]. □

The braid group has an action on the vector space $H_2(B_\lambda, \mathbb{C})$ via the connecting isomorphisms that are induced by the loops in $\mathbb{L}$. This gives a natural map

$$\phi : \mathsf{Br} \cong \mathsf{Aut}(H_2(\mathbb{B}_\lambda, \mathbb{C})) \cong \mathsf{GL}_n(\mathbb{C})$$

The image of this map is called the *monodromy group*.

If we look at the point $\lambda = (-|G| + 1, 1, \ldots, 1)$ we know that the fiber is homeomorphic to the resolution of the Kleinian singularity by theorem 6.5. The 2^{nd} homology has a distinguished basis generated by the n projective lines in the resolutions, which are spheres. The intersection diagram of these spheres is a Dynkin diagram and we can use this to define a bilinear form on H_2. This gives a Weyl group action on H_2.

Theorem 6.19. *The monodromy group is equal to the Weyl group and $\sigma_i \in \mathsf{Br}$ is mapped to to the reflection $s_i \in \mathsf{W}$ associated to the i^{th} node.*

In other words the monodromy group is the Weyl group or Coxeter group of the corresponding Dynkin diagram.

Proof. We illustrate this phenomenon in the A_n case for zero-dimensional fibers, i.e when we look at $f(Z) = (Z - \mu_0) \cdots (Z - \mu_n)$ instead of $f(X, Y, Z) = XY - (Z - \mu_0) \cdots (Z - \mu_n)$. In this case $f^{-1} = \{\mu_0, \ldots, \mu_n\}$, which can be seen as the union of 0-dimensional spheres $SS_i = \{\mu_i, \mu_{i+1}\}$. The transposition $(ii + 1)$ swaps μ_i and μ_{i+1} so SS_i changes orientation and becomes $-SS_i$. The sphere SS_{i-1} becomes $\{\mu_{i-1}, \mu_{i+1}\}$ which is $SS_{i-1} + SS_i$. Similarly the sphere SS_{i+1} becomes $\{\mu_i, \mu_{i+2}\}$ which is $SS_{i+1} + SS_i$. The other spheres don't change so this is precisely the action of the element s_i in the Weyl group.

For the full version in 2-dimensions we refer to Arnol'd's book [?] □

6.5 Summary

Given a hypersurface singularity we have seen two ways to make it smooth.

First of all one can try to deform it and this gives rise to the notion of a versal deformation, which is a parameter space $\mathbb{C}^\mu$ that classifies all deformations of the singularity. In this space we can look at the subspace of all parameters that give smooth deformations.

This subspace is a connected space with a nontrivial fundamental group. This implies that all smooth deformations are homeomorphic, but nontrivial paths give rise to nontrivial homeomorphisms. To describe this we can look at the action of the fundamental group on the homology of a chosen smooth deformation. This gives a map from the fundamental group to the Automorphism group of the homology and its image is called the monodromy group.

A second way of smoothening is resolving a singularity. There are many ways to do this but a particularly interesting one is by looking at moduli spaces of representations of a noncommutative algebra. The parameter space that governs these moduli spaces is a space of characters $\mathbb{Z}^k$.

Again we can look at the subspace that corresponds to the smooth moduli spaces. This is the complement of some hyperplanes in $\mathbb{Z}^k$ and there is a group of reflections associated to these hyperplanes. This group is called the Weyl group.

In the case of Kleinian singularities, these two constructions are related. To each resolution we can associate a deformation and if we complexify $\mathbb{Z}^k$ we can also see this as a deformation space. The Weyl group acts on this space and identifies isomorphic deformations. If we divide out this action we get the versal deformation.

$$\mathbb{Z}^k \otimes \mathbb{C}/\mathsf{W} = \mathbb{C}^\mu$$

Finally, the monodromy group can be identified with the Weyl group.